On the Power Allocation Problem in the Gaussian Interference Channel with Proportional Rate Constraints

Kandasamy Illanko, Alagan Anpalagan, Ekram Hossain, and Dimitrios Androutsos

***Abstract*—This paper takes an analytical approach to solving the optimization problem of finding the power allocation that maximizes the sum-rate of the Gaussian interference channel with any linear power (interference) constraint and proportional rate constraints. It is proved that the sum-rate of the Gaussian interference channel restricted to proportional rate constraints does not have a critical point and the maximum sum-rate subject to said constraints occurs at the boundary of the domain formed by the plane representing the linear power constraint. This is accomplished by using analytic geometry in higher dimensions to show that the curve of intersection of the sum-rate and the proportional rate constraints is always increasing, and intersects the boundary plane representing the linear power constraint at a unique point. A polynomial time (in the number of users) centralized algorithm that finds this point of optimal power allocation is proposed. This is a significant improvement over existing algorithms for related power allocation problems which have exponential time complexity in the number of users. Two distributed algorithms with linear and constant complexities are also presented. Simulation results supporting the analysis and demonstrating the performances of the algorithms are presented.**

***Index Terms*—Gaussian interference channel, sum-rate, power allocation, optimization, fairness, proportional rate constraints, analytic geometry.**

I. Introduction

GAUSSIAN interference channel (GIC) has been used to model the uplinks of code-division multiple access (CDMA) systems, digital subscriber line (DSL) systems, and more recently, ad-hoc networks, and small cell networks such as femtocell networks. Almost 50 years after the first investigation by Shannon [1], an exact description of the capacity region of the interference channel that covers all values of the channel parameters continues to elude us [1]–[4]. Researchers concerned with power allocation that maximizes the system capacity have sidestepped this difficulty by optimizing what is called the sum-rate. This sum-rate is obtained by applying Shannon's original formula for capacity to each user separately while considering the interference from the other users as noise. Even then, the problem of finding the power allocation that maximizes the sum-rate has remained a difficult problem.

In fact, the early works in power allocation on GIC did not consider maximizing the sum-rate. They focused on each user achieving its target SINR and offered distributed solutions based on fixed point algorithms [5], [6]. More recently, strategic game theory has been used to find a competitive equilibrium among users who attempt to maximize their utility functions [7]–[10]. A user's utility function typically includes its transmission rate and a cost function. Game theory approach also offers distributed solutions based on fixed point algorithms or their variants. It is important to note that the sum utility or the sum-rate is not maximized in the game theory based solutions. Also, individual power constraints are used instead of total power constraints.

The difficulty in determining the power allocation that maximizes the sum-rate of the GIC arises due to the fact that the convex structure of the sum-rate is unknown. The only theoretical insight into this problem we have is that the power allocation that maximizes the sum-rate under a total power constraint occurs at the boundary plane formed by the power constraint. This was first proved by Vucic *et al.* [11]. We arrived at the same conclusion, independently, using a different technique [12]. For the problem of maximizing the sum-rate with any fairness or quality-of-service (QoS) constraints, no study has been reported that exploits the structure of the feasible set. However, many search algorithms have been proposed to determine the optimal power allocation under various constraints. Oh *et al.* [13] model the uplink of a CDMA-based cellular system as a GIC and consider the sum-rate maximization problem under the minimum individual signal-to-interference-plus-noise ratio (SINR), total interference, and individual power constraints. They, then, propose a systematic search that finds the optimal power allocation in $O(t^N)$ computations, where N is the number of mobile users and t is the resolution of the power levels. Abadpour *et al.* [14] report that the technique proposed by Oh *et al.* often produces a power allocation that is unfair to some users. To rectify this Abadapour *et al.* introduce maximum individual SINR constraints and propose another algorithm.

Dai *et al.* [15] consider the problem of maximizing the minimum uplink rate of mobile users with individual power constraints as well as minimum and maximum rate constraints.

Manuscript received May 30, 2013; revised September 28, 2013; accepted November 17, 2013. The associate editor coordinating the review of this paper and approving it for publication was L. Song.

This work was supported in part by the National Science and Engineering Research Council (NSERC) of Canada through Discovery Grants and in part by the NSERC Strategic Project Grant STPGP 430285.

K. Illanko, A. Anpalagan, and D. Androutsos are with the Department of Electrical and Computer Engineering, Ryerson University, Toronto, Canada (e-mail: killanko@rnet.ryerson.ca, {alagan, dimitri}@ee.ryerson.ca).

E. Hossain is with the Department of Electrical and Computer Engineering, University of Manitoba, Winnipeg, Canada (e-mail: Ekram.Hossain@umanitoba.ca).

Digital Object Identifier 10.1109/TWC.2013.010214.131970

They use Majorization theory to reduce this optimization problem into a search problem for a real number in a closed interval. Gjendemsj *et al.* [16] propose a suboptimal solution to the power allocation problem that maximizes the system throughput with individual power constraints. This solution is based on an extension of the solution of the two-user case. Qian *et al.* [17] consider the problem of maximizing the weighted sum-rate subject to individual minimum rate and power constraints. They transform this problem into multiplicative linear fractional programming and propose an algorithm that constructs a sequence of polyblocks of decreasing size and searches their vertexes for the optimal solution. Julian *et al.* [18] approximate the Shannon's formula $\log_2(1+\mathrm{SINR})$ with $\log_2(\mathrm{SINR})$ to transform the throughput maximization problem into a geometric program.

The solutions proposed by the work above are sometimes of limited use in the wireless communications systems because of the following reasons. The SINR balancing and game theory algorithms do not maximize the system throughput or sum-rate. All the algorithms work with individual power constraints and not with a total power constraint. Total power constraint is important in practical wireless systems to limit the interference to the neighboring systems. Dividing a total power constraint into equal individual power constraints cannot be efficient. None of the algorithms that maximize the sum-rate uses any knowledge of the structure of the optimization problem. Because of this they fall into the category of search algorithms which are computationally expensive. The complexity of any of these algorithms is equal to or higher than that of Oh *et al.*'s algorithm [13], which, as has been mentioned earlier, is exponential in the number of users ($O(t^N)$).

This paper takes an analytical approach to obtain the solution of the power allocation problem that maximizes the sum-rate of the GIC under any linear power constraint and proportional rate constraints. The goal is to produce an algorithm with a complexity that is practical (polynomial). Proportional rate constraints makes more sense than minimum rate constraints from both the QoS provisioning and fairness (among users) point of views. Proportional rate constraints can be also mapped to proportional delay constraints (in case of saturated traffic scenarios). Furthermore, since existing (minimum rate constraint) algorithms have exponential complexity, a polynomial time solution with a slightly different QoS constraint can be useful. For example, suppose we have a situation where we have to satisfy minimum rate demands and the demands are feasible. The optimum solution using the existing work takes exponential time. However, we can take the ratios of the minimum rates and use our proportional rate algorithm to obtain a near optimal solution in polynomial time. This is worthwhile because the difference between the exponential and polynomial time is extremely large.

The optimization problem undertaken in this paper is challenging because of two reasons. The first is that the objective function is not concave. This excludes many conventional methods from convex analysis such as dual methods and KKT conditions [19]. This difficulty is further exasperated by the second reason that the proportional rate constraints are non-linear. Because of this non-linearity, the maximum sum-rate subject to the proportional rate and linear power constraints could occur at any point in the domain and not necessarily on the power constraint plane (which is the case with minimum rate constraints). Furthermore, in general, the curve of intersection of the proportional rate constraints and the sum could intersect the power constraint plane at more than one point, as illustrated by the following example.

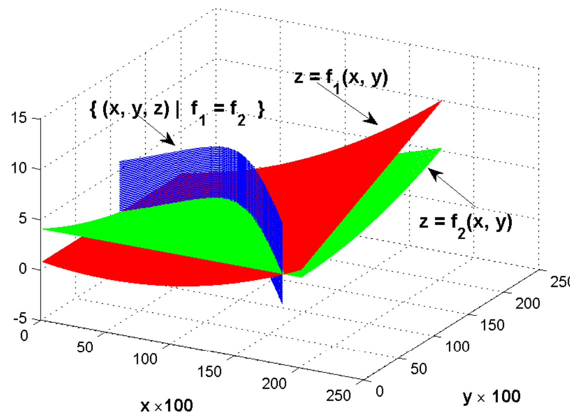


Fig. 1. The locus of $f_1 : f_2 = 1 : 1$.

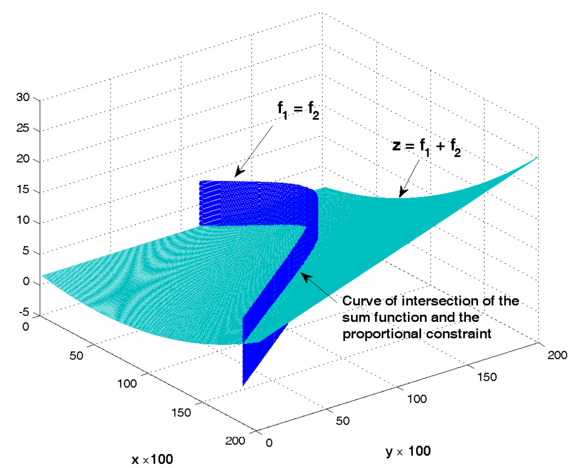


Fig. 2. The curve **C** of intersection of $f_1 : f_2 = 1 : 1$ and the sum.

Example: Consider the sum $f = f_1 + f_2$ of two functions $f_1(x, y) = 2x^2 + 2xy - 2x + 3/4$ and $f_2(x, y) = y^2 + 2xy - 4y + 4$. Suppose the proportional constraint we are interested is 1:1 and the linear constraint is $x + y = 2$. Figures 1-3 show that the curve of intersection of the proportional constraint $f_1 : f_2 = 1 : 1$ and the surface $z = f_1 + f_2$ of the sum intersects the linear constraint plane $x + y = 2$ at two points.

Analytically, the equations $f_1 = f_2$ and $x + y = 2$ can be simultaneously solved to obtain the (x, y) coordinates of these two points. They turn out to be $(1/2, 3/2)$ and $(3/2, 1/2)$.

In our case however, we prove that the curve of intersection of the proportional rate constraints and the sum-rate is always increasing and intersects the power constraint plane at a unique point. Therefore, this point is the maximum.

The contributions of this paper are as follows:

1) The proof that the power allocation that maximizes

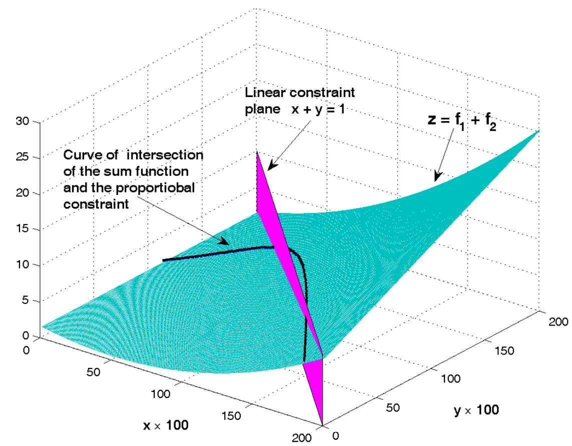


Fig. 3. Intersection of **C** and $x + y = 2$.

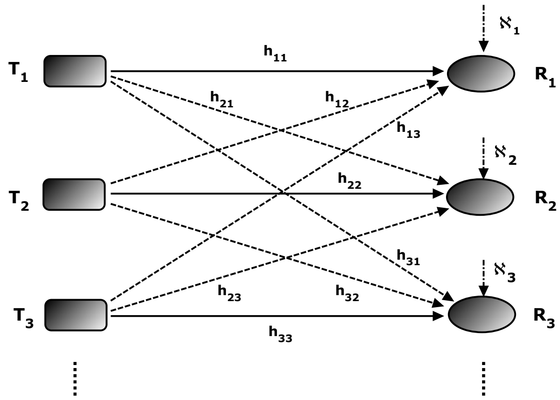


Fig. 4. The N-user Gaussian interference channel.

the sum-rate under the proportional rate and any linear power constraints occurs at a unique point on the power constraint plane.

2) A fast, simple, and stable centralized algorithm that finds this point in polynomial time (specifically, the time complexity of the algorithm is $O(N^3)$).
3) A distributed algorithm that converges to this point with linear time complexity, but does not require any communication between the users, or a user and a central controller.
4) A distributed algorithm with constant time complexity that conforms to the *Shared Memory* model [20] in distributed decision making.

The rest of the paper is organized as follows. Section II presents the system model and the problem formulation. Analysis of the optimization problem and the summary of the analysis can be found in Section III. A centralized algorithm that solves the optimization problem is presented in Section IV, and two distributed algorithms are developed in Section V. Numerical results obtained through simulations are provided in Section VI, followed by conclusion in Section VII. A brief introduction to cross products in dimensions higher than three is given in **Appendix A**.

II. System Model and the Problem Statement

Consider the N-user GIC shown in Fig. 4. User i employs transmitter i to communicate with receiver i but receiver i experiences interference from all other transmitters. h_{ij} denotes the channel gain between transmitter j and receiver i, and p_i denotes the power used by transmitter i. The channel gains are assumed to remain constant in the time period in which the power allocation algorithm is applied. The transmission rate R_i of user i is given by

$$R_i = \log_2\left(1 + \frac{h_{ii}p_i}{\aleph_i + \sum_{j\neq i} h_{ij}p_j}\right), \tag{1}$$

where $\aleph_i$ stands for the additive white Gaussian noise. Letting $N_i = \aleph_i / h_{ii}$ and $a_{ij} = h_{ij}/h_{ii}$,

$$R_i = \log_2\left(1 + \frac{p_i}{N_i + \sum_{j\neq i} a_{ij}p_j}\right). \tag{2}$$

The sum-rate R of the N-user GIC is

$$R = \sum_{i=1}^{N} R_i. \tag{3}$$

Our objective is to solve the optimization problem that determines the power allocation that maximizes the sum-rate R subject to two constraints. The first one is a linear constraint in the transmit powers as follows:

$$\sum_{i=1}^{N} g_i p_i \leq P.$$

This constraint could arise, for example, as an interference constraint in ad-hoc or sensor networks. g_i is the channel coefficient from transmitter i to the interference measuring point. P represents the acceptable interference threshold. If all the g_i's are equal to 1, then this could represent a total power constraint.

The second constraint deals with proportional transmission rates

$$R_1 : R_2 : R_3 : ...R_N = \beta_1 : \beta_2 : \beta_3 : ...\beta_N, \tag{4}$$

where β_i's are non-zero positive real numbers. Letting $\alpha_i = \beta_i/\beta_1$ for $i = 2, 3, 4, ..., N$, the latter can be re-written as the following $N - 1$ equations:

$$\alpha_i R_1 = R_i \quad \text{for } i = 2, 3, 4, ..., N. \tag{5}$$

Often in practice, there may be a total power as well as an interference constraint. This however, does not mean that we have to formulate the optimization problem with both constraints. Suppose the solution to the problem with total power constraint is P^* and the solution to the problem with the interference constraint is P^{**}. Then as we prove in **Appendix B**, the solution to the problem with both constraints is the one among these two points that is closest to the origin.

III. Analysis of the Optimization Problem

In this section, we interpret the objective and the constraints of our optimization problem as hyper surfaces in the $N+1$-dimensional Euclidean space and use analytic geometry in

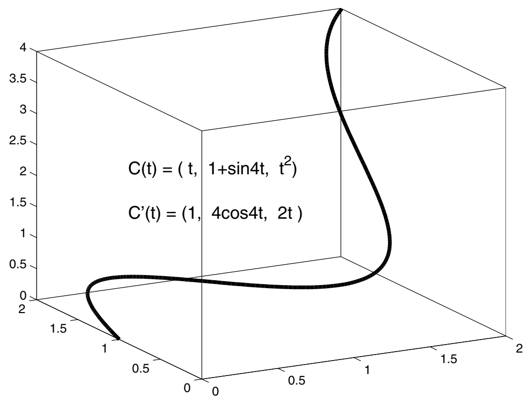


Fig. 5. An increasing space curve.

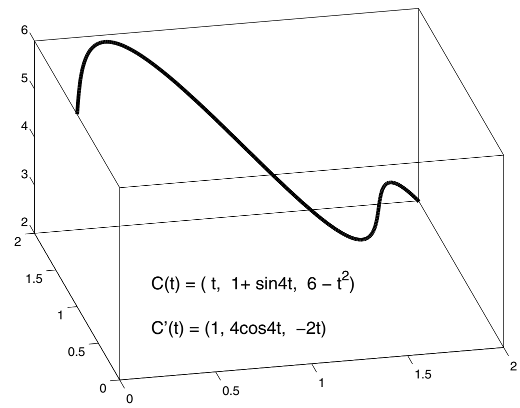


Fig. 6. A decreasing space curve.

higher dimensions to prove that the solution to the optimization problem exists, is unique, and it occurs on the hyper plane formed by the linear power constraint.

A. Objective and Constraints as Hyper Surfaces

Consider first the two-user case and the 3-dimensional Euclidean space with the Cartesian system of coordinates. We can plot p_1 and p_2 along the first two axes and the sum-rate $R(p_1, p_2)$ along the third axis. In this way, $R = R(p_1, p_2)$ will form a 2-dimensional surface in the 3-dimensional space. In this case, there will be only one proportional rate constraint, $\alpha_2 R_1(p_1, p_2) - R_2(p_1, p_2) = 0$, and it will also form a 2-dimensional surface in the 3-dimensional space. Since the third variable R is absent from the rate constraint equation, this latter surface will rise parallel to the R axis and will intersect the surface formed by the sum-rate $R(p_1, p_2)$ in a space curve. In summary, the intersection of two 2-dimensional surfaces in the 3-dimensional space is forming a space curve.

In the case of three users, we need 4 dimensions. The sum-rate $R(p_1, p_2, p_3)$ will form a 3-dimensional hyper surface in the 4-dimensional space. There will be two rate constraints, each forming a 3-dimensional surface. The intersection of the three 3-dimensional surfaces in the 4-dimensional space will form a hyper space curve.

Consider now the N-user case and the $N + 1$ dimensional space. We can plot the powers p_i's along the first N axes and the sum-rate $R(p_1, p_2, ..., p_N)$ along the last axis. In this way, in the $N+1$ dimensional space, R will form an N-dimensional hyper surface. Each of the $N - 1$ proportional rate constraints in (5) will form an N-dimensional surface. The intersection of all of these surfaces - the sum-rate and the constraints - a total of N, N-dimensional surfaces in the $N + 1$-dimensional space will form a hyper space curve $\mathbf{C}$.

B. Methodology of Analysis

We first wish to prove that this space curve $\mathbf{C}$ is always increasing, and therefore, the maximum sum-rate restricted to this curve can only occur on the boundary plane $\sum_i g_i p_i = P$. We then prove that this curve $\mathbf{C}$ indeed intersects the boundary hyper plane $\sum_i g_i p_i = P$, and that the point of intersection is unique.

A single-variable function $f : R \longrightarrow R$ is said to be increasing if $f(a) > f(b)$ whenever $a > b$. We say a space curve $\mathbf{c}(t)$ in N-dimensions defined by the parametric form $\{x_1(t), x_2(t), ..., x_n(t)\}$ is increasing if the single variable function $x_n(t)$ is increasing. It immediately follows that a space curve $\mathbf{c}(t)$ is increasing if the N-th dimensional component $x'_n(t)$ of its tangent vector $\mathbf{c}'(t)$ is positive. Two examples of space curves in 3-dimensions, one increasing and the other decreasing are shown in Fig. 5 and Fig. 6, respectively.

The goal is to prove that the space curve $\mathbf{C}$ is increasing by finding its tangent vector. Since $\mathbf{C}$ is the intersection of N, N-dimensional surfaces, a tangent vector to $\mathbf{C}$ can be found by crossing the N normal vectors of the surfaces. For this purpose, we will use an extension of the familiar cross product of two vectors in three dimensions. An outline of this cross product in higher dimensions is given in **Appendix A**. The orientation of the tangent vector obtained this way would depend on the way we orient the N normal vectors, and the order we cross or write them in the determinant. This is further complicated by the fact that when we evaluate the determinant we need to consider whether N is odd or even. We circumvent these difficulties by first concentrating on the magnitude of the tangent vector and then establishing the orientation using an indirect argument.

C. Normal Vectors of the Surfaces

We start by finding the normal vectors of the N surfaces, namely, the sum-rate surface and the $N - 1$ surfaces from the proportional rate constraints. This can be done by rewriting the equations (3) and (5) as follows:

$$F = \sum_{i=1}^{N} R_i - R = 0,$$
$$H_i = \alpha_i R_1 - R_i = 0 \quad \text{for } i = 2, 3, 4, ..., N, \tag{6}$$

and finding the gradients. Before proceeding, we introduce the notation R_{ij} for the partial derivative of user rate R_i with respect to power p_j. That is, $R_{ij} = \frac{\partial R_i}{\partial p_j}$. The normal vector

$$\mathbf{T} = \begin{vmatrix} \theta_1 & \theta_2 & \dots & \theta_N & \theta_{N+1} \\ \sum_i R_{i1} & \sum_i R_{i2} & \dots & \sum_i R_{iN} & -1 \\ \alpha_2 R_{11} - R_{21} & \alpha_2 R_{12} - R_{22} & \dots & \alpha_2 R_{1N} - R_{2N} & 0 \\ \alpha_3 R_{11} - R_{31} & \alpha_3 R_{12} - R_{32} & \dots & \alpha_3 R_{1N} - R_{3N} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_N R_{11} - R_{N1} & \alpha_N R_{12} - R_{N2} & \dots & \alpha_N R_{1N} - R_{NN} & 0 \end{vmatrix}. \tag{9}$$

$$t_{N+1} = \pm \begin{vmatrix} \sum_i R_{i1} & \sum_i R_{i2} & \dots & \sum_i R_{iN} \\ \alpha_2 R_{11} - R_{21} & \alpha_2 R_{12} - R_{22} & \dots & \alpha_2 R_{1N} - R_{2N} \\ \alpha_3 R_{11} - R_{31} & \alpha_3 R_{12} - R_{32} & \dots & \alpha_3 R_{1N} - R_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_N R_{11} - R_{N1} & \alpha_N R_{12} - R_{N2} & \dots & \alpha_N R_{1N} - R_{NN} \end{vmatrix} = \pm|\mathbf{M}_0|. \tag{10}$$

of the sum-rate surface is

$$\nabla F = \left(\sum_i R_{i1}\,,\ \sum_i R_{i2}\,,\ \sum_i R_{i3},\ ...,\sum_i R_{iN}\,,\ -1 \right). \tag{7}$$

The normal vectors of the $N-1$ surfaces in (6) are

$$\nabla H_i = (\alpha_i R_{11} - R_{i1},\, \alpha_i R_{12} - R_{i2}, ..., \alpha_i R_{1N} - R_{iN},\, 0)\,, \quad \text{for } i = 2,3,4,...,N. \tag{8}$$

D. Tangent Vector to Curve $\mathbf{C}$ *and its Last Component*

Crossing the N vectors in (7) and (8), we obtain the tangent vector $\mathbf{T}$ of the curve of intersection $\mathbf{C}$ of the sum-rate and the rate constraints, shown in (9), where $\theta_1, \theta_2, \theta_3, \dots, \theta_N, \theta_{N+1}$ are the unit vectors along the $N+1$ coordinate directions. The $N+1$-dimensional component of $\mathbf{T}$, which we denote by t_{N+1}, is shown in (10). The determinant in (10) can be shown to be row equivalent to the determinant below:

$$\begin{vmatrix} R_{11} & R_{12} & R_{13} & \dots R_{1N} \\ R_{21} & R_{22} & R_{23} & \dots R_{2N} \\ R_{31} & R_{32} & R_{33} & \dots R_{3N} \\ \vdots & \vdots & \vdots & \ddots \vdots \\ R_{N1} & R_{N2} & R_{N3} & \dots R_{NN} \end{vmatrix}. \tag{11}$$

From (2), the partial derivatives are calculated as

$$R_{ij} = \begin{cases} \frac{1}{p_i + I_i}, & \text{if } j = i, \\ -\frac{a_{ij} p_i}{(p_i + I_i)(I_i)}, & \text{if } j \neq i, \end{cases}$$

where

$$I_i = N_i + \sum_{j \neq i} a_{ij} p_j. \tag{12}$$

Substituting in (11), we obtain:

$$\begin{vmatrix} \frac{1}{p_1+I_1} & \frac{-a_{12}p_1}{I_1(p_1+I_1)} & \frac{-a_{13}p_1}{I_1(p_1+I_1)} & \cdots & \frac{-a_{1N}p_1}{I_1(p_1+I_1)} \\ \frac{-a_{21}p_2}{I_2(p_2+I_2)} & \frac{1}{p_2+I_2} & \frac{-a_{23}p_2}{I_2(p_2+I_2)} & \cdots & \frac{-a_{2N}p_2}{I_2(p_2+I_2)} \\ \frac{-a_{31}p_3}{I_3(p_3+I_3)} & \frac{-a_{32}p_3}{I_3(p_3+I_3)} & \frac{1}{p_3+I_3} & \cdots & \frac{-a_{3N}p_3}{I_3(p_3+I_3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-a_{N1}p_N}{I_N(p_N+I_N)} & \frac{-a_{N2}p_N}{I_N(p_N+I_N)} & \frac{-a_{N3}p_N}{I_N(p_N+I_N)} & \cdots & \frac{1}{p_N+I_N} \end{vmatrix}.$$

At this point, we emphasize that none of the powers p_i's can be zero, for if any of the p_i's is zero, then that user's rate would be zero, and that would violate our condition that none of the β_i is zero in (4). It can be then shown that the matrix of the determinant above is equivalent to the matrix $\mathbf{M}$ below:

$$\mathbf{M} = \begin{bmatrix} I_1 & -a_{12}p_2 & -a_{13}p_3 & \cdots & -a_{1N}p_N \\ -a_{21}p_1 & I_2 & -a_{23}p_3 & \cdots & -a_{2N}p_N \\ -a_{31}p_1 & -a_{32}p_2 & I_3 & \cdots & -a_{3N}p_N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{N1}p_1 & -a_{N2}p_2 & -a_{N3}p_3 & \cdots & I_N \end{bmatrix}. \tag{13}$$

From (12), since $I_i = N_i + \sum_{j\neq i} a_{ij}p_j$, $I_i > \sum_{j\neq i} a_{ij}p_j, \quad \forall i$. This shows that the matrix $\mathbf{M}$ satisfies the condition

$$|m_{ii}| > \sum_{j \neq i} |m_{ij}|, \quad \forall i,$$

and therefore, is *diagonally dominant.* The determinant of a *diagonally dominant* matrix cannot be zero [21]. Hence $\det(\mathbf{M}) \neq 0$, and by extension, $t_{N+1} \neq 0$. Since t_{N+1} is clearly continuous in p_i's, by intermediate value theorem, t_{N+1} must be either always positive or always negative.

E. Curve C is Increasing

We now consider the special case where $\beta_i = 1$ for all i and the users all experience the same channel conditions. That is, $a_{ik} = a_{jk}$ for all i, j, k. It is clear that in this case the rate constraint (4) would imply that the powers are all equal. Letting $p_i = p$ for all i in (2),

$$\frac{dR_i}{dp} = \frac{N_i}{(N_i + p\sum_{j\neq i} a_{ij})^2 + p(N_i + p\sum_{j\neq i} a_{ij})} \ > \ 0.$$

Hence, $\frac{dR}{dp} > 0$. This shows that the tangent line to the curve of intersection of the sum-rate and the rate constraint points in the direction of increasing R. In other words, $t_{N+1} > 0$, in this special case. Combining this with the result that t_{N+1} is either always positive or always negative, we conclude that t_{N+1} is always positive.

The fact that t_{N+1} is always positive implies that the curve of intersection of the sum-rate and the rate constraints is always increasing and the maximum sum-rate restricted to this curve $\mathbf{C}$ can only occur at the boundary plane $\sum_{i=1}^{N} g_i p_i = P$.

F. Curve $\mathbf{C}$ *Intersects the Power Constraint Plane at a Unique Point*

We now focus on showing that the curve $\mathbf{C}$ does indeed intersect the plane $\sum_{i=1}^{N} g_i p_i = P$, and that the point of

$$\mathbf{T}.\mathbf{n} = \begin{vmatrix} g_1 & g_2 & g_3 & \dots & g_N & 0 \\ \sum_i R_{i1} & \sum_i R_{i2} & \sum_i R_{i3} & \dots & \sum_i R_{iN} & -1 \\ \alpha_2 R_{11} - R_{21} & \alpha_2 R_{12} - R_{22} & \alpha_2 R_{13} - R_{23} & \dots & \alpha_2 R_{1N} - R_{2N} & 0 \\ \alpha_3 R_{11} - R_{31} & \alpha_3 R_{12} - R_{32} & \alpha_3 R_{13} - R_{33} & \dots & \alpha_3 R_{1N} - R_{3N} & 0 \\ \alpha_4 R_{11} - R_{41} & \alpha_4 R_{12} - R_{42} & \alpha_4 R_{13} - R_{43} & \dots & \alpha_4 R_{1N} - R_{4N} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_N R_{11} - R_{N1} & \alpha_N R_{12} - R_{N2} & \alpha_N R_{13} - R_{N3} & \dots & \alpha_N R_{1N} - R_{NN} & 0 \end{vmatrix}. \tag{14}$$

$$\mathbf{T}.\mathbf{n} = \pm g_1 \begin{vmatrix} S_{22} - \gamma_2 S_{21} & S_{23} - \gamma_3 S_{21} & \dots & S_{2N} - \gamma_N S_{21} \\ S_{32} - \gamma_2 S_{31} & S_{33} - \gamma_3 S_{31} & \dots & S_{3N} - \gamma_N S_{31} \\ S_{42} - \gamma_2 S_{41} & S_{43} - \gamma_3 S_{41} & \dots & S_{4N} - \gamma_N S_{41} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N2} - \gamma_2 S_{N1} & S_{N3} - \gamma_3 S_{N1} & \dots & S_{NN} - \gamma_N S_{N1} \end{vmatrix}. \tag{15}$$

intersection is unique. This is accomplished by proving that $\mathbf{C}$ never becomes parallel to $\sum_{i=1}^N g_i p_i = P$. Note that a normal vector to $\sum_{i=1}^N g_i p_i = P$ is $\mathbf{n} = (g_1, g_2, g_3, \dots g_N, 0)$. For $\mathbf{C}$ to become parallel to $\sum_{i=1}^N g_i p_i = P$, the dot product $\mathbf{T}.\mathbf{n}$ must vanish.

Writing $\mathbf{T}.\mathbf{n}$ explicitly, in (14), and expanding by the last column and letting $S_{ij} = \alpha_i R_{1j} - R_{ij}$, we obtain

$$\mathbf{T}.\mathbf{n} = \pm \begin{vmatrix} g_1 & g_2 & g_3 & \dots & g_N \\ S_{21} & S_{22} & S_{23} & \dots & S_{2N} \\ S_{31} & S_{32} & S_{33} & \dots & S_{3N} \\ S_{41} & S_{42} & S_{43} & \dots & S_{4N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & S_{N3} & \dots & S_{NN} \end{vmatrix}.$$

Letting $\gamma_j = g_j / g_1$, taking away γ_j times the first column from column j, and expanding by the first row, we arrive at (15). At this point we go back to (13) and observe that $\mathbf{M}$ is a Z-matrix that is diagonally dominant. Therefore, $\mathbf{M}$ must be positive definite [22]. By extension, the matrix $\mathbf{M_0}$ in the determinant that defined t_{N+1} in (10) must be either positive or negative definite. Using our earlier shorthand $S_{ij} = \alpha_i R_{1j} - R_{ij}$, matrix $\mathbf{M_0}$ can be re-written as

$$\mathbf{M_0} = \begin{bmatrix} \sum_i R_{i1} & \sum_i R_{i2} & \sum_i R_{i3} & \dots & \sum_i R_{iN} \\ S_{21} & S_{22} & S_{23} & \dots & S_{2N} \\ S_{31} & S_{32} & S_{33} & \dots & S_{3N} \\ S_{41} & S_{42} & S_{43} & \dots & S_{4N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & S_{N3} & \dots & S_{NN} \end{bmatrix}.$$

This matrix can be shown to be equivalent to the matrix $\mathbf{M_2}$ shown in (16). Note that $\mathbf{M_2}$ must also be either positive or negative definite.

We now notice that the determinant in (15) is a principal minor of the matrix $\mathbf{M_2}$ above. Since $\mathbf{M_2}$ is either positive or negative definite, any principal minor of $\mathbf{M_2}$ must be non-zero. Hence, $\mathbf{T}.\mathbf{n} \neq 0$. This shows that $\mathbf{C}$ never becomes parallel to $\sum_{i=1}^N g_i p_i = P$. Note that $\mathbf{C}$ cannot intersect any of the coordinate planes because such an intersection would force one of the p_i's to be zero and that would violate our condition that none of the β_i is zero. This proves $\mathbf{C}$ must intersect $\sum_{i=1}^N g_i p_i = P$.

Now suppose that $\mathbf{C}$ intersects $\sum_{i=1}^N g_i p_i = P$ at two points. Then, after the first intersection, $\mathbf{C}$ must turn back towards $\sum_{i=1}^N g_i p_i = P$ at some point. At this turning point, $\mathbf{C}$ must become parallel to $\sum_{i=1}^N g_i p_i = P$. This is a contradiction.

We now summarize the result of our analysis in the theorem below.

Theorem 1: The power allocation that maximizes the sum-rate R of the GIC subject to the constraint $\sum_{i=1}^N g_i p_i \leq P$ and the proportional rate constraints in (5) exists, unique, and lies at the hyper plane $\sum_{i=1}^N g_i p_i = P$.

IV. Centralized Algorithm for Power Allocation

In the last section, we proved that the power allocation that maximizes the sum-rate of the GIC subject to proportional rate constraints and $\sum_{i=1}^N g_i p_i \leq P$ lies on a unique point in the plane $\sum_{i=1}^N g_i p_i = P$. In this section, we first develop a polynomial time algorithm that finds this optimal point. Then we compare the complexity of this algorithm to a typical power allocation algorithm for GIC.

The straightforward way of determining the power levels at the optimal point is to solve the $N-1$ equations corresponding to the proportional rate constraints with the equation for power constraint $\sum_{i=1}^N g_i p_i = P$. The $N-1$ equations for the proportional rate constraints are non-linear. This means we have to solve a system of non-linear equations. The most popular method to solve a system of non-linear equations is the multi-variable Newton-Raphson method. However, this method is highly unstable. That is, it is extremely sensitive to the initial guess for the solution, and depending on this initial point, may not converge at all. Because of this, we abandon this approach and investigate other ways to find the solution. We propose the following method which is guaranteed to converge to the globally optimal solution as well as faster than the multi-variable Newton-Raphson method.

A. Algorithm

We decompose the task at hand into two problems, for each of which there is a "mature technology" [19] available to solve that problem. We first write the proportional rate equations in terms of User-1's rate R_1 to obtain a linear system in the powers p_i's. We then substitute the "solution" of this system

$$\mathbf{M_2} = \begin{bmatrix} S_{22} - \gamma_2 S_{21} & S_{23} - \gamma_3 S_{21} & \dots & S_{2N} - \gamma_N S_{21} & S_{21} \\ S_{32} - \gamma_2 S_{31} & S_{33} - \gamma_3 S_{31} & \dots & S_{3N} - \gamma_N S_{31} & S_{31} \\ S_{42} - \gamma_2 S_{41} & S_{43} - \gamma_3 S_{41} & \dots & S_{4N} - \gamma_N S_{41} & S_{41} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{N2} - \gamma_2 S_{N1} & S_{N3} - \gamma_3 S_{N1} & \dots & S_{NN} - \gamma_N S_{N1} & S_{N1} \\ \sum_i R_{i2} - \gamma_2 \sum_i R_{i1} & \gamma_3 \sum_i R_{i3} - \sum_i R_{i1} & \dots & \sum_i R_{iN} - \gamma_N \sum_i R_{i1} & \sum_i R_{i1} \end{bmatrix}. \tag{16}$$

into the power constraint equation to form a single non-linear equation in R_1.

If we let

$$S_i = 2^{R_i} - 1 = 2^{\alpha_i R_1} - 1,$$

(1) can be re-written as

$$h_{ii} p_i - S_i \sum_{j \neq i} h_{ij} p_j = S_i \aleph_i \quad \text{for } i = 1, 2, 3, ..., N. \tag{17}$$

Note that this is a linear system of N equations in the solution variables p_i. If we know the channel coefficients and the α_i's, then the coefficient matrix $\mathbf{A}$ of this linear system can be written in terms of R_1. If $\mathbf{q}$ is the column vector with entries $S_i \aleph_i$, this linear system can be written as

$$\mathbf{A}\mathbf{p} = \mathbf{q}. \tag{18}$$

Using Cramer's Rule, we can write p_i's in terms of $\mathbf{A}$ and $\mathbf{q}$. In other words, the p_i's can be written in terms of R_1. Suppose $\mathbf{A}_i$ is the matrix obtained by replacing the ith column of $\mathbf{A}$ by the column vector $\mathbf{q}$. Then Cramer's Rule gives

$$p_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}. \tag{19}$$

Substituting in the power constraint equation $\sum_{i=1}^{N} g_i p_i = P$ gives us

$$\sum_i g_i \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})} = P. \tag{20}$$

Since the entries of $\mathbf{A}_i$ and $\mathbf{A}$ can be explicitly written in terms of R_1, the above is a non-linear equation in R_1. After solving this equation for R_1, we can go back to (19) or to (18) and obtain the power values.

We now present our low-complexity algorithm that finds the power allocation that maximizes the sum-rate of the GIC subject to proportional rate constraints as well as a linear constraint on the transmit powers.

Centralized Algorithm

1) Solve the non-linear equation (20) and obtain R_1.
2) Solve the linear system (18) to obtain the power levels.

Note that the convergence of this algorithm to the global optimal point is guaranteed because of the following. **Theorem 1** states that for a particular P and a particular set of proportional rate constraints, there exists only a unique set of power levels and those power levels actually globally maximize the sum-rate. This unique set of power levels corresponds to a unique R_1 as given by (1). This shows that R_1 exists and is unique. Therefore, the non-linear solver will find R_1 in Step 1 of the algorithm. Because of the proportional rate constraints now the algorithm knows the unique R_i's of all users. The unique set of R_i's implies a unique set of proportional rate constraints. This and a particular P give rise to only one set of power levels by **Theorem 1**. Therefore, the linear system in Step 2 has a unique solution and the linear system solver will find it.

B. Complexity

The first step of the Centralized Algorithm consists of solving a single non-linear equation that finds the rate of the first user. The second step involves solving a system of N by N linear equations that gives the powers, where N is the number users. We know that the time it takes to solve a single non-liner equation is negligible compared to the time it takes to solve a linear system of equations. Therefore, as far as the complexity is concerned we can ignore the first step of our algorithm. More importantly, the time spent in the first step is independent of the number of users N.

Solving linear systems is considered a matured technology, meaning there are extremely reliable software packages that can solve systems with a very large number of variables very accurately within a very short time [19]. Since the time complexity of solving an N by N system is $O(N^3)$ [19], the complexity of our algorithm is $O(N^3)$, where N is the number of users.

Let us compare this to the best algorithm in the literature that maximizes the sum-rate, that is Oh *et al.*'s algorithm [13]. It is important to note here that the Oh algorithm works with minimum rate constraints and our algorithm works with proportional rate constraints. Therefore, a direct comparison may not be fair, and one has to be careful. Oh algorithm has a complexity of $O(t^N)$, where t is the number of resolution levels and N is the number of users. Note that this is an algorithm with exponential time complexity. The time it takes to find the optimal solution not only increases at an exponential rate in the number of users, but it also increases with the resolution levels. One might argue that the minimum rate constraints are tougher to satisfy and therefore the comparison is not fair. However, the difference between polynomial and exponential time is so huge that even after making an allowance for the minimum rate constraints our algorithm will stand out. In order to obtain a clear picture about the relative speed of the two algorithms let us consider a concrete example.

Suppose we have 10 users and the power allocation levels are 0 to 10 mW in 1 mW intervals. This means $t = 10$. The time taken by Oh algorithm is approximately 10^{10}, whereas our's takes only 10^3, which is an improvement by a factor of 10^7. In Section VI we demonstrate the efficacy of our algorithm in solving even the sum-rate maximization problem with *minimum rate constraints*.

V. Distributed Algorithms for Power Allocation

The Centralized Algorithm presented in the last section requires a duplex control channel between each user and a central controller. Each user must estimate its channel coefficient and report it to the central controller via this channel and then obtain the optimum power level from the central controller. In this section, we provide two distributed algorithms, one of which completely eliminates the need for this control channel and the other eliminates the need for a central controller. It should be noted that even though the structure and the design of these algorithms seem fairly intuitive on their own, the proof that these algorithms actually converge to the power allocation that maximizes the sum-rate while maintaining the proportional rate constraints depends on **Theorem 1** of Section III.

A. Distributed Algorithm-1

This algorithm assumes no communication between the users, or between a user and a central controller. The only requirement is that the users have access to synchronized clocks and that each user is aware of the total interference constraint P. The time period allocated to power control during the control part of the frame is divided into time slots and the power levels are updated on each time slot until convergence.

The total power or the interference constraint places considerable limitations on the possible distributed algorithms. Since there is no way to check if this constraint is met in the middle of the algorithm, the only choice is to start with a power allocation that satisfies this constraint and then preserve this during each iteration. Hence, the algorithm starts with User-i employing a power $p_i = P/(Ng_i)$.

After this, all users update their powers at the aforementioned synchronized time slots. During each time slot only one user increases its power by s/g_i, while all the others decrease their power by $s/[g_i(N-1)]$, where s is a predetermined step size. Note that this keeps the sum interference at P. Which user needs to increase its power is determined in the following way. Suppose γ_i is the SINR of User-i. We define *Fairness Quotient* FQ_i of User-i as follows:

$$FQ_i = (1+\gamma_i)^{1/\alpha_i}.$$

The proportional rate constraints in (5) can be now re-written as

$$FQ_i = K \quad \text{for } i = 1, 2, 3, ..., N,$$

where K is a number that is independent of i and α_1 is defined as unity. By **Theorem 1**, for a fixed interference constraint P, there is only one power allocation and hence one K that satisfies this equation. This implies that for a fixed interference constraint P, there is only one K. To emphasize this one-to-one relationship between K and P, we re-write the last equation as

$$FQ_i = K_P \quad \text{for } i = 1, 2, 3, ..., N.$$

Note that the above equation will be valid only at the optimal power allocation. For an arbitrary power allocation satisfying the interference constraint, such as at the beginning of the proposed distributed algorithm, some FQ_i's will be lower than K_P while the others greater than K_P. Note also that while the value of K_P can be calculated using the Centralized Algorithm when all users' channel gains are known, in the current distributed decision making scenario, users have no way of determining K_P. However, each user can calculate its own current Fairness Quotient from its SINR. The proposed distributed algorithm works by allowing the user with the lowest FQ_i value to increase its power in each time slot. At the beginning of each time slot, each user starts a timer that expires after cFQ_i time units, where c is an appropriate constant that was previously agreed upon. The user whose timer expires first increases its power by s/g_i. At this point, other users will observe a reduction in their SINR. This is the signal for these users to reduce their power by $s/[g_i(N-1)]$, stop their timers, and wait for the beginning of the next time slot.

At the beginning of the algorithm, a larger step size s will be useful so that the gap in FQ values can be bridged sooner. However, as can be seen from the proof of the **Theorem 2** below, at the end of the algorithm, a larger value of s will make the FQ values oscillate with a large amplitude. A step size that decreases during the run of the algorithm would be the best one; and for this reason, a function $f(t)$, which we call the accelerating factor, is introduced, where t is the number of iterations. $f(t)$ must be an increasing function in t; but exactly what function to choose is best decided by simulations. The algorithm is terminated when the change in the SINR in successive iterations becomes too small to be of any practical value, that is, when it is smaller than a predetermined number γ_{stop}.

Distributed Algorithm-1

---

1) $p_i := P/(Ng_i)$, $\forall i$.
2) Compute $FQ_i = (1+\gamma_i)^{1/\alpha_i}$ and at the beginning of the next time slot set the timer to expire exactly after cFQ_i time units.
3) The User-i whose timer expires first increases its power by s/g_i.
4) If any user's SINR (γ) decreases before its timer expires, it decreases its power by $s/[g_i(N-1)]$.
5) Go to step 2 with $s := s/f(t)$ unless change in SINR is smaller than γ_{stop}.

The following Lemmas and definitions are necessary for the proof that this distributed algorithm converges.

Lemma 1: For a given interference constraint P, there cannot be any power distribution that makes $FQ_i > K_P$ for all i.

Proof: Suppose there is such a power distribution. For ease of explanation, let us consider the two-user case first. Without loss of generality, we can assume there is a power distribution such that $FQ_1 > FQ_2 > K_P$. By incrementally reducing User-1's power but increasing the power of User-2, we can make $FQ_1 = FQ_2 > K_P$, while satisfying the interference constraint P. This would imply that there are two different power distributions both satisfying a particular proportional rate and interference constraints. This contradicts **Theorem 1**. The argument for the case with more than two users follows the same path. The contrary assumption implies that by a re-distribution of power levels we can make $FQ_1 = FQ_2 = FQ_3 = ... = FQ_N > K_P$ while satisfying the interference constraint P. This contradicts **Theorem 1**. ∎

Definition: *Increases to within* δ.
A sequence $\{a_i\}$ is said to be increasing to within δ if $a_{i+1} > a_i - \delta$ for all i, where δ is a fixed positive number whose magnitude is small compared to any a_i.

Note that *Deccreases to within* δ can be defined in a similar manner.

Definition: *Converges to* C *within* δ *from below*.
A sequence $\{a_i\}$ is said to converge to C within δ from below if there exists an M such that $C - \delta \le a_i \le C$ for $i > M$.

Converges to C *within* δ *from above* is defined in a similar way.

Lemma 2: A sequence that is bounded above (below) and increases (decreases) to within δ converges to its least upper bound (greatest lower bound) within δ from below (above).

Proof: Follows from the well-known theorem that a sequence that is bounded above (below) and increases (decreases) converges to its least upper bound (greatest lower bound). ∎

Theorem 2: Distributed Algorithm-1 converges to the unique, optimum power levels mentioned in **Theorem 1**.

Proof: The behavior of the algorithm with a fixed step size s is established first. In other words, assume $f(t) = 1$ for now. Consider two sequences of numbers selected from the FQ values from successive iterations of the algorithm. First one, which we call the *Lower Sequence* consists of the smallest FQ value during each iteration. Suppose during the first iteration User-3 has the lowest FQ value, during the second iteration User-1 has the lowest FQ value and during the third iteration User-5 has the lowest FQ value and so on. With a slight abuse of notation, the lower sequence would then look like $FQ_3, FQ_1, FQ_5, \cdots$.

At first it would appear that the *Lower Sequence* is always increasing. This is because at any iteration, the user with the lowest FQ is allowed to increase its power and hence its FQ. However, a closer examination shows that this is true only at the beginning. Sooner or later these increasing values will cross over the FQ value of a user who is decreasing its power. The new lowest FQ may not be necessarily higher than the previous one. As can be seen from Fig. 7, the new FQ value can be lower than the old FQ value.

When a user decreases its power its FQ value will decrease accordingly. How much FQ decreases will depend on the channel gains as well as the current power levels. Since the changes in power levels are the same from iteration to iteration, the magnitude of the change in FQ for a particular User-i will remain approximately the same throughout the run of the algorithm. However, it will be different from user to user. Let ΔFQ_M stand for the magnitude of the maximum change in the FQ over all users. That is, let

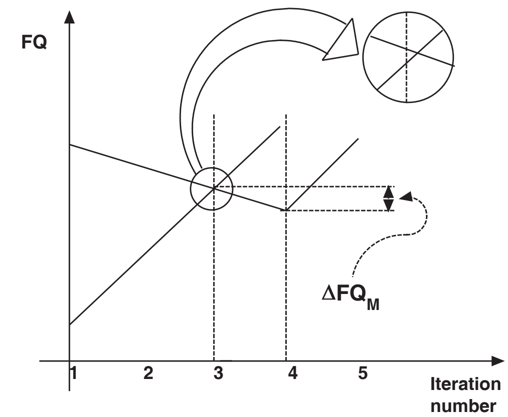


Fig. 7. Worst-case crossing.

$$\Delta FQ_M = \max_i \Delta FQ_i.$$

As illustrated in Fig. 7, the new FQ value can be lower than the old FQ value by at most ΔFQ_M. This shows that the *Lower Sequence* is increasing to within ΔFQ_M.

The *Lower Sequence* gets arbitrarily close to K but cannot exceed K. It cannot exceed K because of the following reason. Suppose at the end of an iteration the *Lower Sequence* exceeds K. This would imply that the FQ values at the end of this iteration are all greater than K. This contradicts **Lemma 1**. Thus K is the least upper bound of the *Lower Sequence* which increases to within ΔFQ_M. By **Lemma 2**, it converges to K within ΔFQ_M from below.

Now consider the *Upper Sequence* which consists of the highest FQ values on successive iteration. It can be shown that the upper sequence decreases to within ΔFQ_M and gets arbitrarily close to K but never goes below K. Hence, the *Upper Sequence* converges to K, to within ΔFQ_M from above.

The sequence of FQ_i values of any User-i through the iterations is trapped between the Lower and Upper sequences and hence will eventually oscillate about K_P to within $2\Delta FQ_M$.

The accelerating factor $f(t)$ will make the effective step size approach zero as the iteration number increases. This will have the effect of making ΔFQ_M approach zero. Thus, the FQ_i values will all eventually approach K_P.

Earlier, we explained that **Theorem 1** implied that the power levels will satisfy $FQ_i = K_P$ for all i only at the global optimal point. Therefore, Distributed Algorithm-1 converges to the global optimum power allocation. ∎

B. Distributed Algorithm-2

An algorithm that conforms to the *Shared Memory* model in the study of distributed algorithms [20] is presented here. The proposed algorithm assumes that control channels are available between each user and a shared memory location so that it can write its rate and obtain the rates of others or the sum-rate. Note that this memory can be provided by one of the users, a *leader*.

This algorithm also starts with power levels that satisfy the interference constraint. Unlike the last algorithm, all users simultaneously update their power levels during each time slot. However, this is done in a way that the interference constraint is met throughout the run of the algorithm. The change in power level is based on the each user's proportion of its rate to the sum-rate of all N users. This proportional rate is defined as

$$r_i = \frac{R_i}{\sum_i R_i}. \tag{21}$$

At the beginning of each time slot the sum-rate of the N users is obtained by each user from the shared memory. This enables each user to calculate its proportional rate at each iteration of the algorithm. Let r_{it} be the proportional rate of User-i at the beginning of iteration number t and r_i be the proportional rate requested by User-i. During iteration number t, User-i would change its power by

$$\Delta p_i = (k/g_i)(r_i - r_{it}), \tag{22}$$

where k is an appropriate constant. Note that this means a user whose current proportional rate is lower than its required proportional rate will end up increasing its power while a user whose current proportional rate is higher than its required proportional rate will end up decreasing its power. The following Lemma proves that the proposed update in power levels satisfies the interference constraint.

Lemma 3: Suppose the power levels of the users satisfy the interference constraint $\sum_i g_i p_i = P$ at the beginning of an iteration. The power levels after a change of power levels given by $\Delta p_i = (k/g_i)(r_i - r_{it})$ will still satisfy the same interference constraint.

Proof: Since r_i and r_{it} are ratios of user rates to the sum-rate as defined in (21), we have

$$\sum_i r_i = \sum_i r_{it} = 1.$$

The change in the sum $g_i p_i$ is

$$\begin{aligned}\Delta \sum_i g_i p_i &= \sum_i g_i \Delta p_i \\ &= \sum_i k(r_{ir} - r_{it}), \text{ using (22)} \\ &= k \sum_i (r_{ir} - r_{it}) \\ &= k\left(\sum_i r_i - \sum_i r_{it}\right) = k(1-1) = 0.\end{aligned}$$

■

Distributed Algorithm-2

1) $p_i := P/(Ng_i)$, $\forall i$.
2) $p_i := p_i + \Delta p_i = p_i + (k/g_i)(r_i - r_{it})$, $\forall i$.
3) Go to step 2 unless change in SINR is smaller than γ_{stop}.

The following theorem proves that Distributed Algorithm-2 converges as long as the proportionality constant k is not too large. It should be noted here that the condition on k is nothing new. For example, the gradient decent algorithm, which finds the location of the minimum of f, increments its independent variables in step sizes of $k\nabla f$, and will overshoot and oscillate about the minimum point if k is too large.

Theorem 3: Distributed Algorithm-2 converges to the unique optimal point mentioned in **Theorem 1** where the proportional rate and the interference constraints are satisfied.

Proof: Consider User-i on iteration number t. Without loosing generality we may assume that its proportional rate r_{it} at this point is lower than what it requested. This means it will increase its power. Since everyone whose proportional rate is smaller than their requested proportional rate would have increased their powers while the others lowered their powers, the new proportional rate of User-i would be higher. As long as k is not too large, it will still be smaller than the requested proportional rate r_i. This shows that the proportional rates r_{it}s will form an increasing sequence whose least upper bound is r_i. Hence, it will converge to r_i. This shows that the users' rates will converge to the proportional rates requested.

Lemma 3 showed that during every iteration of the algorithm the power levels remain on the interference constraint plane. Therefore, the point of convergence is on this plane. By **Theorem 1**, there is only one point on this plane that satisfies the proportional rate constraints. Hence, this point of convergence is the unique optimal point. ■

C. Complexity

Let Δp_i be the difference between the initial and final power level of User-i during the execution of Distributed Algorithm-1. In the worst case, this user may have to decrease its power throughout the run of the algorithm. In each iteration, it will decrease the power by $s/[(N-1)g_i]$, or approximately by $s/[(Ng_i]$, where s is the step size. If s is the average step size during the run of the algorithm, then it would take $\Delta p_i N g_i/s$ iterations.

Each iteration consists of one time slot of fixed duration in which all users update their powers, even though they have to wait until their timers expire before they can increase the power or notice a decrease in their SINR and reduce the power. The important question is whether the length of the time slot should depend on the number of users. The time slot should be long enough to allow each user to estimate its SINR and calculate $FQ_i = (1 + \text{SINR}_i)^{1/\alpha}$. It should also be long enough to accommodate a length of time equal to $c[FQ_i]_{\max}$. But $[FQ_i]_{\max}$ does not depend on how many users are there and c can be chosen appropriately to fit $[FQ_i]_{\max}$ into an appropriate time interval in which users can estimate their SINR and set the timers. This shows the length of the time slot does not depend on the number of users. If we denote the length of each time slot by τ and $\lambda_1 = \{\Delta p_i g_i\}_{\max}$, then the complexity order of Distributed Algorithm-1 would be $\tau\lambda_1 N/s$.

A similar calculation shows that the time complexity order of Distributed Algorithm-2 is $\tau\lambda_2/s$, where λ_2 is a constant. Its run time is independent of the number of users.

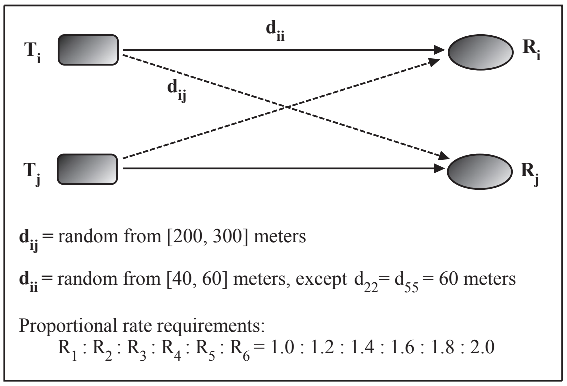


Fig. 8. Scenario-1: Simulation details (only 2 of the 6 transceivers are shown).

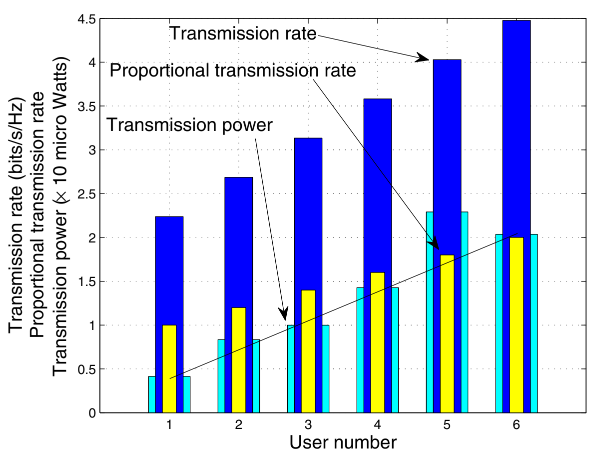


Fig. 9. Transmission rates and power levels for Scenario-1.

VI. Numerical Results

Numerical results obtained using simulations for three different scenarios are presented here. In each scenario there are 6 users. A path-loss coefficient of 4 is used to calculate the channel coefficients from the distances. The performance of the Centralized Algorithm is demonstrated first followed by that of the distributed algorithms.

A. Centralized Algorithm

The first scenario is about power allocation in GIC under a total power constraint. The details are shown in Fig. 8, where we deliberately place the 2nd and 5th users in unenviable positions: their transmitter to receiver distances are a bit greater than the other users. The power allocation by the Centralized Algorithm with the total power constraint of 80 micro Watts is illustrated in Fig. 9. Not surprisingly, all the runs of the Centralized Algorithm produce transmission rates that are exactly at the proportional rates requested: 1.0000, 1.2000, 1.4000, 1.6000, 1.8000 and 2.000. We use bar charts to illustrate the patterns in power allocation that is required to produce this rate ratios. Note that in Fig. 9, the power levels of User-2 and User-5 are higher than what should be expected (the inclined line) for their rate demands. This should be anticipated because, by our design, their transmitter to receiver distances are greater than the others.

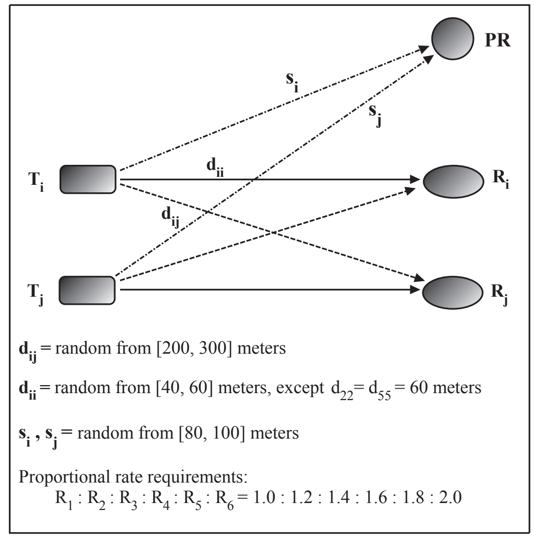


Fig. 10. Scenario-2: Simulation details (only two of the 6 transceivers are shown).

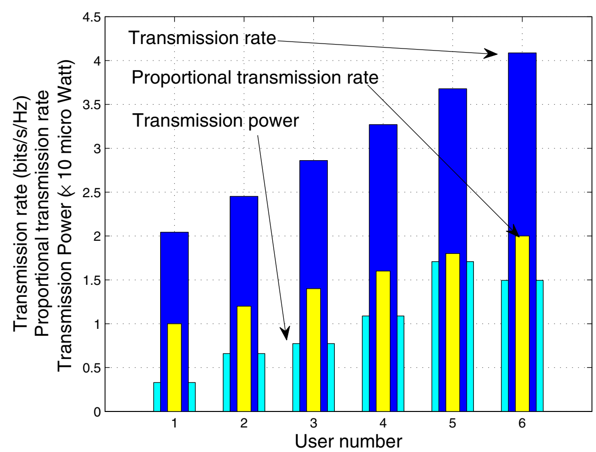


Fig. 11. Transmission rates and power levels for Scenario-2.

The second and third scenarios deal with power allocation in GIC under an interference constraint. Details of Scenario-2 are shown in Fig. 10, where the 6 users from the previous scenario now become transmitting users, and an interference measuring point, indicated by PR, is added. An interference tolerance level of $P = 1$ pico Watts is used. The power levels prescribed by our algorithm are shown in Fig. 11. We notice that the rates of users for Scenario-2 are smaller compared to those for Scenario-1. This is because an interference tolerance level of 1 pico Watt and the assumed distances between the transmitter and interference measuring points in Scenario-2 put a more stringent condition on the power levels than a total power constraint of 80 micro Watts in Scenario-1.

The third scenario is identical to the second one except that the value of s_5 is fixed at 70 meters, which is lower than the other s_i's. We put User-5 in a tough position by placing it closer to the interference measuring point than the others. The power allocation for this Scenario is shown in Fig. 12. We notice that the rates of all users are smaller compared to those in Scenario-2. User-5's proximity to the interference

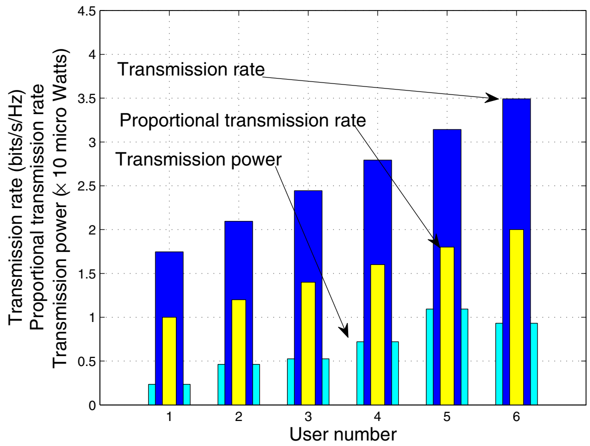


Fig. 12. Transmission rates and power levels for Scenario-3.

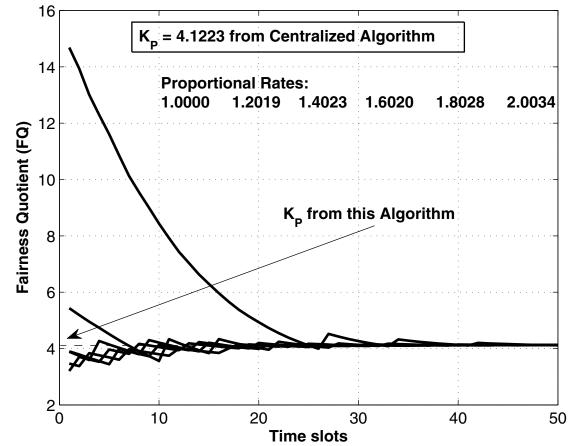


Fig. 13. Distributed Algorithm-1: Convergence of the Fairness Quotients.

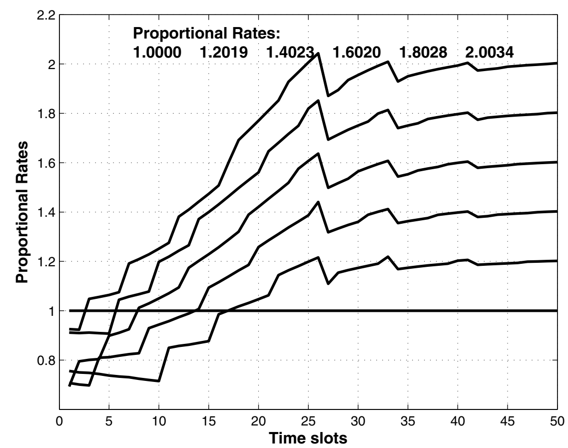


Fig. 14. Distributed Algorithm-1: Convergence of the proportional rates.

measuring point pushes its power down, which in turn has the effect of lowering the powers of all the other users through the proportional rate constraint.

The Oh algorithm [13] mentioned before works with minimum rate constraints but our algorithm works with proportional rate constraints. Therefore, it is not possible to compare our algorithm to Oh algorithm directly. However, we demonstrate the performance of our algorithm in solving the problem for which Oh algorithm was designed. For this, we consider the sum-rate maximization problem with minimum individual rate constraints and a total power constraint. We calculate the proportional rates using the minimum rate demands and use those in our algorithm. If the minimum rates are feasible then our algorithm should produce a result that satisfies the minimum rate demands. Note that our algorithm will not provide optimal sum-rate. This is because it uses that exact proportional rates whereas minimum rates give more leeway in the feasible set. But we expect our algorithm to be much faster than Oh algorithm. The channel gains are obtained from Scenario-1, but we restrict the number of users to 3 because of the long execution time for Oh algorithm.

Table I shows that our algorithm satisfies the minimum rate demands and achieves above 90% of the sum-rate achieved by Oh algorithm but it is faster by a factor of about 100. This is with three users and according to the complexity analysis in the paper, this should increase rapidly with more users. For example, with 10 users, our algorithm should be $100 \times (3^3/10^3)/[(10)^3/(10)^{10}]$ times, or 10^7 times faster.

B. Distributed Algorithms

The second scenario in the previous subsection with the same interference tolerance level of $P = 1$ pico Watt is used to evaluate the distributed algorithms. Fig. 13 shows the behavior of the Distributed Algorithm-1 with an accelerating factor of $f(t) = t^{1/30}$. To show that the algorithm behaves exactly as predicted in the convergence proof, we plot the FQ values of the users during the execution of the algorithm. In the proof of convergence, it was mentioned that a particular FQ curve will jump in the opposite direction when it crosses another curve. This can be clearly seen in Fig. 13. The proportional rates produced by this algorithm, shown in Fig. 13, are satisfactory.

For the purpose of comparison, we use the Centralized Algorithm to calculate the K_P value and it turns out to be 4.1223. The convergence proof of the Distributed Algorithm-1 claims that the FQ values should converge to K_P. In Fig. 13, we see the FQ values converging to a number just above 4. A separate plot in Fig. 14 shows how the proportional rates converge to the required ratios.

The behavior of the Distributed Algorithm-2 for the same simulation scenario is shown in Fig. 15. We see a much faster convergence to the required proportional rates. It should be noted that this performance gain is obtained at the cost of a communication channel between each user and a shared memory.

VII. Conclusion

We have presented a deterministic solution to the optimization problem of finding the power allocation that maximizes the sum-rate of the Gaussian interference channel subject to any linear power constraint and proportional rate constraints. This has been accomplished using analytic geometry in higher

TABLE I
CENTRALIZED ALGORITHM COMPARED WITH OH ALGORITHM

Minimum rate demand (bits/s/Hz)		Sum-rate (bits/s/Hz)	User-1 rate (bits/s/Hz)	User-2 rate (bits/s/Hz)	User-3 rate (bits/s/Hz)	Time (Sec.)
4, 4, 4	Oh	14.31	5.06	4.39	4.86	14.88
	Ours	14.29	4.76	4.76	4.76	0.14
6, 3, 3	Oh	13.68	6.00	3.65	4.03	14.69
	Ours	12.72	6.36	3.18	3.18	0.14
6, 4, 3	Oh	13.44	6.01	4.04	3.39	14.72
	Ours	13.19	6.09	4.06	3.04	0.16
3, 3, 6	Oh	13.01	3.85	3.16	6.01	14.76
	Ours	12.34	3.08	3.08	6.17	0.14
3, 5, 3	Oh	14.11	4.51	5.00	4.60	14.86
	Ours	12.72	3.47	5.78	3.47	0.15
2, 6, 2	Oh	11.35	2.76	6.01	2.57	14.73
	Ours	10.23	2.05	6.14	2.05	0.13

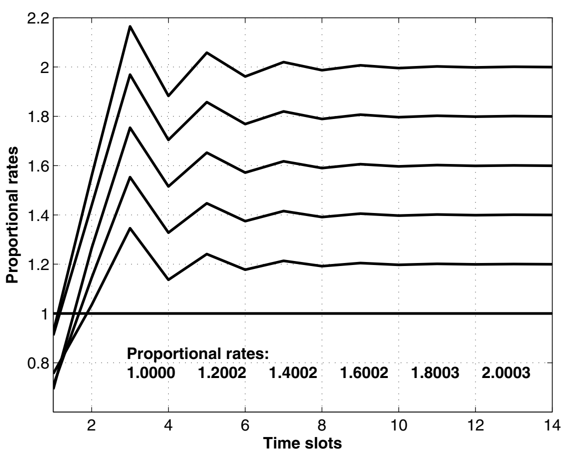


Fig. 15. Distributed Algorithm-2: Convergence of the proportional rates.

dimensions to show that the curve of intersection of the sum-rate and the proportional rate constraints is always increasing and the maximum sum-rate occurs at the unique point where this curve intersects the boundary plane formed by the linear power constraint. A polynomial time centralized algorithm as well as two distributed algorithms that find the optimal power allocation have been proposed. Simulation results supporting the analysis and demonstrating the performances of the algorithms have been presented.

APPENDIX A
CROSS PRODUCT IN HIGHER DIMENSIONS

In 3-dimensions, two non-parallel vectors can be used to define a unique direction that is perpendicular to both vectors, using the familiar cross product. In higher dimensions, the idea of a cross product of two vectors falls apart because of the following reason. In 4-dimensions, given two non-parallel vectors, there is an infinite number of vectors that are perpendicular to both these vectors. In fact, these vectors will form a plane that is perpendicular to the given two vectors. Suppose we use X, Y, Z, W to label the axes in 4-dimensions. Consider the X and Y axes. Z axis is perpendicular to both X and Y, but so is W. In fact, every direction in the ZW-plane is perpendicular to both X and Y directions. Similarly, in 5-dimensions, the vectors perpendicular to given two vectors will form a 3-dimensional subspace. In light of this, mathematicians have defined various advanced "products" of two vectors in higher spaces such as Wedge product and Clifford product.

While Wedge product and Clifford product have found numerous important applications in Physics and Engineering, it turns out that for our present work, we could use the familiar cross product after all. There is a way to extend the cross product into higher dimensions [23]. In 4-dimensions, the idea is to use 3 linearly independent vectors to define a unique direction perpendicular to these three. For example, W axis is the only direction perpendicular to X, Y, and Z in 4-dimensions. Moreover, this new direction can be found by using the familiar determinant style formulation.

Let us consider two illustrative examples from the 4-dimensional space. Suppose we use $\theta_1, \theta_2, \theta_3$, and θ_4 to denote the unit vectors in the 4 coordinate directions. Let us cross $(1,0,0,0)$, $(0,1,0,0)$, and $(0,0,1,0)$ together. Note that these are the unit vectors in the coordinate directions X, Y and Z. We expect the result to be the unit vector in the W direction. Please note that in the following, the result is obtained simply by calculating the determinant.

$$\begin{vmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (0,0,0,1),$$

as expected. As a second example, consider the cross product of $\mathbf{v_1} = (1,1,0,0), \mathbf{v_2} = (0,1,1,0)$ and $\mathbf{v_3} = (0,0,1,1)$.

$$\begin{vmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = (1,-1,1,-1) = \mathbf{v}.$$

We can easily verify that the result $\mathbf{v}$ is perpendicular to each of the $\mathbf{v_i}$'s by taking the dot product. That is, $\mathbf{v}.\mathbf{v_i} = 0$ for

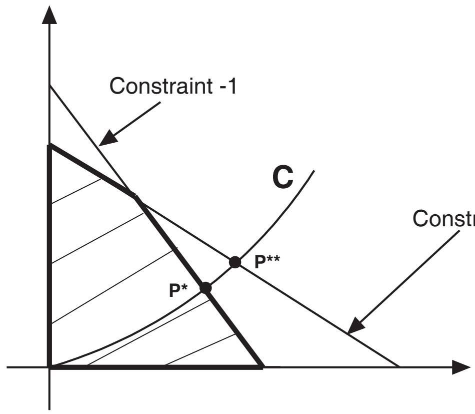


Fig. 16. Solution with two linear constraints.

$i = 1, 2, 3$.

In general, in the N-dimensional space, we can use $N - 1$ linearly independent vectors to define a unique direction that is perpendicular to all $N - 1$ vectors, using the same determinant formulation of the extension of the 3-dimensional cross product.

APPENDIX B
TWO LINEAR CONSTRAINTS ON THE POWERS

Proposition: Suppose the solution to the optimization problem with total power constraint is P^* and the solution with the interference constraint is P^{**}. Then the solution with both constraints is the one among these two points that is closest to the origin.

Proof: Note that the feasible set with both constraints is the intersection of the feasible sets with each of the constraints alone. Therefore, both points P^* and P^{**} are upper bounds for the new feasible set. By the proof of **Theorem 1**, both lie on the intersections of curve **C** with the respective constraint planes (Fig. 16). When we travel along curve **C** starting from the origin, we will encounter one of these points first. This point is the only one among the two points that is guaranteed to lie on the intersection of the two original feasible sets. This point will be the solution to the new optimization problem, because an upper bound that lies on the feasible set is the maximum. Note that this point has the shortest distance from the origin measured along curve **C**.

Distance along the curve **C** from the origin can be replaced with distance from the origin unless curve **C** "turns too much," or more precisely, unless curve **C** becomes parallel to one of the planes. The proof of **Theorem 1** established that this is not the case. ■

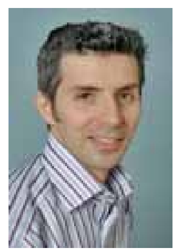
REFERENCES

[1] C. E. Shannon, "Two-way communication channels," in *Proc. 1961 Berkeley Symp. Mathematics, Statistics, Probability*, vol. 1, pp. 611–644.
[2] T. S. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. Inf. Theory*, 1981, vol. IT-27, no. 1, pp. 49–60.
[3] I. Sason, "On achievable rate regions for the Gaussian interference channel," *IEEE Trans. Inf. Theory*, vol. 50, no. 6, pp. 1345–1356, June 2004.
[4] X. Shang, G. Kramer, and B. Chen, "A new outer bound and noisy interference sum-rate capacity for Gaussian interference channels," *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 689–699, Feb. 2009.
[5] J. Zander, "Distributed cochannel interference control in cellular radio systems," *IEEE Trans. Veh. Technol.*, vol. 41, no. 3, Aug. 1992.
[6] G. J. Foschini and Z. Miljanic, "A simple distributed autonomous power control algorithm and its convergence," *IEEE Trans. Veh. Technol.*, vol. 42, no. 4, Nov. 1993.
[7] V. Shah, N. B. Mandayam, and D. J. Goodman, "Power control for wireless data based on utility and price," in *Proc. 1998 IEEE International Symp. Personal, Indoor, Mobile Radio Commun.*, vol. 3, pp. 1427–1432.
[8] C. U. Saraydar, N. B. Mandayam, and D. J. Goodman, "Efficient power control via pricing in wireless data networks," *IEEE Trans. Commun.*, vol. 50, no. 2, pp. 291–303, Feb. 2002.
[9] W. Yu and J. M. Cioffi, "Competitive equilibrium in the Gaussian interference channel," in *Proc. 2000 IEEE ISIT*.
[10] D. T. Ngo, L. B. Le, T. Le-Ngoc, E. Hossain, and D. I. Kim, "Distributed interference management in two-tier CDMA femtocell networks," *IEEE Trans. Wireless Commun.*, vol. 11, no. 3, Mar. 2012.
[11] N. Vucic, S. Shi, and M. Schubert, "DC programming approach for resource allocation in wireless networks," in *Proc. 2010 International Symp. Modeling Optimization Mobile, Ad Hoc Wireless Netw.*
[12] K. Illanko, A. Anpalagan, and D. Androutsos, "Dual methods for power allocation for radios coexisting in unlicensed spectra," in *Proc. 2010 IEEE Global Commun. Conf.*
[13] S. J. Oh and A. C. K. Soong, "QoS-constrained information-theoretic sum capacity of reverse link CDMA systems," *IEEE Trans. Wireless Commun.*, vol. 5, no. 1, pp. 3–7, Jan. 2006.
[14] A. Abadpour, A. S. Alfa, and A. C. K. Soong, "Closed form solution for maximizing the sum capacity of reverse-link CDMA system with rate constraints," *IEEE Trans. Wireless Commun.*, vol. 7, no. 4, pp. 1179–1183, Apr. 2008.
[15] J. Dai, Z. Ye, and X. Xu, "Power allocation for maximizing the minimum rate with QoS constraints," *IEEE Trans. Veh. Technol.*, vol. 58, no. 9, pp. 4989–4996, Nov. 2009.
[16] A. Gjendemsj, D. Gesbert, G. E. Yien, and S. G. Kiani, " Binary power control for sum rate maximization over multiple interfering links," *IEEE Trans. Wireless Commun.*, vol. 7, no. 8, pp. 3164–3173, Aug. 2008.
[17] L. P. Qian, Y. J. Zhang, and J. Huang, "MAPEL: achieving global optimality for a non-convex wireless power control problem," *IEEE Trans. Wireless Commun.,* vol. 8, no. 3, pp. 1553–1563, Mar. 2009.
[18] D. Julian, M. Chiang, D. O'Neill, and S. Boyd, "QoS and fairness constrained convex optimization of resource allocation for wireless cellular and ad hoc networks," in *Proc. 2002 IEEE INFOCOM*.
[19] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.
[20] N. Lynch, *Distributed Algorithms*. Morgan Kaufmann Publishers, 1996.
[21] O. Taussky, "A recurring theorem on determinants," *American Mathematical Monthly,* vol. 56, no. 10, pp. 672–674, Dec. 1949.
[22] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1991.
[23] M. Reeder, Boston College, MT210 Linear Algebra Course Notes, Chapter 18, "A Tour of Four Dimensions." Available: https://www2.bc.edu/~reederma/Linalg18.pdf

Kandasamy Illanko obtained his B.Sc. with Honors in Electrical and Electronic Engineering from the University of Peradeniya, Sri Lanka, in 1985, following which he worked as an Assistant Lecturer in the Department of Electrical and Electronic Engineering at the University of Peradeniya. In 1987, he was awarded a commonwealth scholarship to pursue graduate studies at the University of Toronto. After completing his M.A.Sc. in Computer Engineering at the University of Toronto in 1991, Illanko worked in the industry, mainly in system level programming in the C language/Unix environment. Currently he is a Ph.D. candidate in the Department of Electrical and Computer Engineering at the Ryerson University, Toronto, Canada. Illanko's research interests include applications of non-convex optimization and game theory to resource allocation in cognitive radios and energy-efficient transmissions in wireless communications.

Alagan Anpalagan received his B.A.Sc., M.A.Sc. and Ph.D. degrees in Electrical Engineering from the University of Toronto, Canada. He joined the Department of Electrical and Computer Engineering at Ryerson University in 2001 and was promoted to Full Professor in 2010. He served the department as Graduate Program Director (2004-09) and the Interim Electrical Engineering Program Director (2009-10). During his sabbatical (2010-11), he was a Visiting Professor at Asian Institute of Technology and Visiting Researcher at Kyoto University. Dr. Anpalagan's industrial experience includes working at Bell Mobility on 1xRTT system deployment studies (2001), at Nortel Networks on SECORE R&D projects (1997) and at IBM Canada as IIP Intern (1994). Dr. Anpalagan directs a research group working on radio resource management (RRM) and radio access & networking (RAN) areas within the WINCORE Lab at Ryerson. His current research interests include cognitive radio resource allocation and management, wireless cross layer design and optimization, collaborative communication, green communications technologies, machine-to-machine communication, and small cell networks. He is co-author/editor of the books *Small Cell Networks* (Cambridge Press, 2014), *Handbook of Green Information and Communication Systems* (Academic Press, 2012) and *Routing in Opportunistic Networks* (Springer-Verlag, 2013). Dr. Anpalagan has published more than 200 technical papers in his research areas. Dr. Anpalagan serves as an Editor for the IEEE COMMUNICATIONS SURVEYS & TUTORIALS (2012-), IEEE COMMUNICATIONS LETTERS (2010-13) and *Springer Wireless Personal Communications* (2009-), and a past Editor for *EURASIP Journal of Wireless Communications and Networking* (2004-2009). He also served as Gust Editor for special issues in *ACM/Springer MONET Green Cognitive and Cooperative Communication and Networking* (2012), *EURASIP Radio Resource Management in 3G+ Systems* (2006) and *EURASIP Fairness in Radio Resource Management for Wireless Networks* (2008).

Ekram Hossain is a Professor in the Department of Electrical and Computer Engineering at University of Manitoba, Winnipeg, Canada. He received his Ph.D. in Electrical Engineering from University of Victoria, Canada, in 2001. Dr. Hossain's current research interests include design, analysis, and optimization of wireless/mobile communications networks, cognitive radio systems, and network economics. He has authored/edited several books in these areas (http://home.cc.umanitoba.ca/∼hossaina). His research has been widely cited in the literature (more than 6880 citations in Google Scholar with an *h-index* of 42 until November 2013). Dr. Hossain serves as the Editor-in-Chief for the IEEE COMMUNICATIONS SURVEYS AND TUTORIALS and an Editor for the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS - COGNITIVE RADIO SERIES and *IEEE Wireless Communications*. Also, he is a Member of the IEEE Press Editorial Board. Previously, he served as the Area Editor for the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS in the area of "Resource Management and Multiple Access" from 2009-2011, and an Editor for the IEEE TRANSACTIONS ON MOBILE COMPUTING from 2007-2012. Dr. Hossain has won several research awards including the University of Manitoba Merit Award in 2010 (for Research and Scholarly Activities), the 2011 IEEE Communications Society Fred Ellersick Prize Paper Award, and the IEEE Wireless Communications and Networking Conference 2012 (WCNC'12) Best Paper Award. He is a Distinguished Lecturer of the IEEE Communications Society (2012-2015). Dr. Hossain is a registered Professional Engineer in the province of Manitoba, Canada.



Dimitrios Androutsos received his B.A.Sc., M.A.Sc. and Ph.D. degrees from the University of Toronto, Canada in 1992, 1994, and 1999, respectively, all in Electrical and Computer Engineering. He has held numerous industry positions in both the U.S. and Canada and currently is a Full Professor at Ryerson University, Canada in the Department of Electrical and Computer Engineering, where he has served as Interim Department Chair (2009-2010), Program Director of Electrical Engineering (2006-1012) and Industrial Internship Co-Ordinator (2005-2012). Prof. Androutsos also held the NSERC Chair in Design Engineering (2010-12) . He has co-authored more than 100 technical research papers in numerous research areas. His recent active research interests are in the areas of multimedia processing and transmission, stereoscopy and 3-D digital cinema. He is an IEEE Senior Member, Associate Editor of the *Journal of Multimedia*, was Associate Editor of IEEE SIGNAL PROCESSING LETTERS and was a guest editor of *IEEE Signal Processing Magazine*.