Frequency and Power Allocation for Energy Efficient OFDMA Systems with Proportional Rate Constraints

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***Abstract*—A two-step solution to the problem of finding the subchannel and power allocation that maximizes the energy efficiency of the OFDMA based transmissions from a base station, under proportional rate and total power constraints, is presented. A low complexity subchannel assignment is followed by an optimal power allocation that is obtained via a single non-linear equation. The proposed algorithm has the same computational complexity as the best algorithm in the literature for the same problem but with minimum rate constraints. While the convergence of the algorithm in the literature is not guaranteed, the algorithm in this paper is proven to converge. Simulation results show that the proposed algorithm out-performs the one in the literature when the rate constraints are ignored.**

***Index Terms*—OFDMA, energy efficiency.**

I. Introduction

THE need to reduce the global footprint of mobile communications together with the increasing demand for the data rates has necessitated research into the energy efficiency of the transmissions from base stations [1], [2]. Finding the power and frequency allocation that maximize the bits/Joule/Hz energy efficiency (EE) of transmissions has proven to be difficult because EE is not concave in the powers, and the channel allocation in orthogonal frequency division multiple access (OFDMA) is computationally challenging. Parametrized convex programming [3], Charnes-Cooper transformation [4], and bi-level optimization [5], [6] have been proposed to deal with power allocation. Heuristic approaches have been suggested for the channel allocation. However, none of the algorithms proposed to solve the problem has been proven to converge [6]. This raises questions about their utility in practical systems with widely varying channel conditions. It should be noted that there are very efficient algorithms for solving related EE problems in the medium access (MAC) layer [7], but these algorithms do not maximize the EE at the physical layer.

This letter presents a two-step solution to the problem of finding the subchannel and power allocation that maximizes the EE of OFDMA based transmissions from a base station. Proportional rate constraints rather than minimum rate constraints are used because of the following reason. It is known that the power allocation that maximizes the energy efficiency produces relatively low data rates [2]- [7] compared to throughput maximizing power allocation under a total power constraint. This shows that a power allocation procedure that satisfies users minimum rate demands will produce a power allocation that is far from the un-constraint optimal energy efficient power allocation. In other words, forcing the users to accept proportional rates, instead of minimum rate demands will produce transmissions at a higher energy efficiency. Proportional rate constraints can also be mapped in to proportional delay constraints in case of saturated traffic scenarios. The solution to the problem is obtained using an algorithm that is guaranteed to converge. It has the same complexity as the best algorithm in the literature [6] with minimum rate constraints, and out-performs that algorithm in simulations when the rate constraints are ignored.

Section II formulates the optimization problem. The analysis and the solution are presented in Section III, the numerical results in Section IV, and the conclusion in Section V.

II. System Model and the Optimization Problem

Consider the downlink of a single cell with N users and K orthogonal subchannels. Each subchannel is assigned exclusively to one user. If the channel gain on the kth subchannel is a_k, the transmission power p_k, the noise spectral density σ, and $h_k = a_k/\sigma$, then the system EE of the transmissions over all K subchannels in bits/Joule/Hz can be written as:

$$EE = \frac{\sum_{k=1}^{K} \log_2 (1 + h_k p_k)}{p_c + \eta \sum_{k=1}^{K} p_k}, \tag{1}$$

where η is the reciprocal of the efficiency and p_c is the circuit power [6] of the downlink transmitter.

Assume a yet to be determined subchannel assignment protocol is used to distribute the subchannels among the users. Suppose a total of K_1 subchannels - subchannel 1 through subchannel k_1 - are assigned to User-1. A total of K_2 subchannels - subchannel $k_1 + 1$ through subchannel k_2 - are assigned to User-2 and so on. A total of K_n subchannels - subchannel $k_{n-1} + 1$ through subchannel k_n - are assigned to User-n. Then the rate r_n of User-n can be written as:

$$r_n = \sum_{k=k_{n-1}+1}^{k_n} \log_2 (1 + h_k p_k), \tag{2}$$

where $k_0 = 0$. Suppose the rate demand of User-n is α_n times that of User-1. In other words, the proportional rate demands are given by

$$\alpha_{n+1} r_1 - r_{n+1} = 0 \text{ for } n = 1, 2, ..., N - 1. \tag{3}$$

The goal is to maximize the EE in (1) subject to (3) and a total power constraint P_T using a two step optimization procedure - first proposing a subchannel assignment protocol and then solving the following power allocation problem:

$$\begin{aligned} &\max_{p_1, p_2, \cdots p_K} EE \\ &\text{subject to} \\ &\text{C1} : \alpha_{n+1} r_1 - r_{n+1} = 0 \text{ for } n = 1, 2, ..., N - 1. \\ &\text{C2} : P_T - \sum_{k=1}^{K} p_k \geq 0. \end{aligned} \tag{4}$$

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III. Analysis and Solution

The computational complexity of an optimal channel assignment protocol would prevent it from being useful in practice. Because of this, we first present a low complexity channel assignment protocol. Then we analyze the power allocation problem and propose a power allocation procedure.

A. Channel Assignment

It is straight forward to prove that the EE of the single user system is directly proportional to its channel gain. This suggests one should assign each subchannel to the user with the largest gain on that subchannel. However, this may leave some users with no subchannels at all. On the other hand, from a fairness point of view, one might want to assign each user the subchannel for which this user has the greatest channel gain. We take an approach that balances these two perspectives.

The Channel Assignment Protocol

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1. Re-label the users in the descending order of their
   proportional rate demands.
2. while (there are subchannels left to assign)
3.      for each user
4.           assign the subchannel in which it has the
             largest gain.
5.      end for
6.      for each user
7.           Assign a subchannel if that user happens
             to have the largest gain on that subchannel.
8.      end for
9. end while
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B. Power Allocation

Having assigned the subchannels using the protocol given above, we set out to solve the optimization problem in (4). The objective function in (4) is not concave in the powers. However, since the numerator is positive and concave, and the denominator is positive and affine, (4) is a concave fractional program that can be transformed into a concave program using a transformation proposed by Charnes and Cooper [8].

Charnes-Cooper Transformation (CCT): The concave fractional program $\max\{N(\mathbf{x})/D(\mathbf{x})|\ M(\mathbf{x}) \geq 0,\ L(\mathbf{x}) = 0\}$ reduces to the concave program $\max\{tN(\mathbf{y}/t)|\ tM(\mathbf{y}/t) \geq 0,\ L(\mathbf{x}) = 0,\ tD(\mathbf{y}/t) = 1,\ t > 0\}$, under the transformation $t = 1/D(\mathbf{x}),\ \mathbf{x} = \mathbf{y}/t$.

The substitution $p_k = y_k/t$ for all $k, t = 1/(p_c + \eta \sum_k p_k)$ reduces (4) to a standard concave maximization problem:

$$\max_{\mathbf{y},\, t} \quad f(\mathbf{y}, t) = t \sum_{k=1}^{K} \log_2\left(1 + h_k y_k/t\right)$$

subject to

$$\text{C1}: \alpha_{n+1} \sum_{k=1}^{k_1} \log_2\left(1 + h_k y_k/t\right) - \sum_{k=k_n+1}^{k_{n+1}} \log_2\left(1 + h_k y_k/t\right) = 0, \quad \text{for } n = 1, 2, \ldots, N-1.$$

$$\text{C2}: tP_T - \sum_{k=1}^{K} y_k \geq 0 \quad \text{C3}: \eta \sum_{k=1}^{K} y_k + p_c t - 1 = 0,\ t > 0 \tag{5}$$

Henceforth, as the first line above indicates, the objective function will be referred to by f. The Lagrangian is:

$$\begin{aligned} &L(\mathbf{y},\, t, \lambda, \mu) \\ &= f + \lambda\left[\eta \sum_{k=1}^{K} y_k + p_c t - 1\right] + \gamma\left[tP_T - \sum_{k=1}^{K} y_k\right] \\ &\quad + \sum_{n=1}^{N-1} \mu_n \left[\alpha_{n+1} r_1 - r_{n+1}\right], \end{aligned} \tag{6}$$

where λ, γ and μ_n are the dual variables. Since (5) is a standard concave problem, the KKT conditions [9] are necessary and sufficient for optimality. These conditions are listed below, where l is used to denote ln2 and q_k for $\frac{h_k p_k}{1 + h_k p_k}$. The first three equations result from the constraints and the rest from the derivatives.

$$\alpha_{n+1} \sum_{k=1}^{k_1} \log_2\left(1 + h_k p_k\right) = \sum_{k=k_n+1}^{k_{n+1}} \log_2\left(1 + h_k p_k\right), \quad \text{for } n = 1, 2, 3, \ldots, N-1. \tag{7}$$

$$\gamma\left[tP_T - \sum_{k=1}^{K} y_k\right] = 0, \qquad tP_T - \sum_{k=1}^{K} y_k \geq 0, \qquad \gamma \geq 0 \tag{8}$$

$$\eta \sum_{k} y_k + p_c t - 1 = 0 \tag{9}$$

$$\begin{aligned} &\sum_{k=1}^{K} \left(l \log_2(1 + h_k p_k) - q_k\right) - \frac{1}{t} \sum_{n=1}^{N-1} \mu_n \alpha_{n+1} \sum_{k=1}^{k_1} q_k \\ &\quad + \frac{\mu_1}{t} \sum_{k=k_1+1}^{k_2} q_k + \ldots \\ &\quad + \frac{\mu_{N-1}}{t} \sum_{k=k_{N-1}+1}^{K} q_k \ + \lambda l p_c + \gamma l P_T = 0 \end{aligned} \tag{10}$$

$$\left(1 + \frac{1}{t} \sum_{n=1}^{N-1} \mu_n \alpha_{n+1}\right) \frac{q_k}{p_k} + \lambda \eta l - \gamma l = 0, \quad \text{for } k = 1, 2 \ldots, k_1 \tag{11a}$$

$$\left(1 - \frac{\mu_1}{t}\right) \frac{q_k}{p_k} + \lambda \eta l - \gamma l = 0, \quad \text{for } k = k_1 + 1, k_1 + 2 \ldots, k_2 \tag{11b}$$

$$\vdots$$

$$\left(1 - \frac{\mu_{N-1}}{t}\right) \frac{q_k}{p_k} + \lambda \eta l - \gamma l = 0, \quad \text{for } k = k_{N-1} + 1, k_{N-1} + 2, \ldots, N \tag{11c}$$

The equation in (8) suggests two cases: either $\gamma = 0$, or $tP_T - \sum_{k=1}^{K} y_k = 0$. The first implies that the solution of the optimization problem lies inside the power constraint plane $tP_T - \sum_{k=1}^{K} y_k = 0$. The second implies that the solution lies on the power constraint plane.

Case I: $\gamma = 0$

Equations (11a) - - (11c) show that the quantity $p_k + 1/h_k$ remains constant for all subcarriers assigned to a particular user. That is, each user has his own water level. Denoting User-n's water level by w_n,

$$p_k + 1/h_k = w_n, \quad \text{for } k = k_{n-1}+1, k_{n-1}+2, ...n \quad \text{for } n = 1, 2, ...N. \tag{12}$$

Equations (10) - - (12) can be used to show that

$$f = \frac{\sum_{n=1}^{N} \alpha_n}{\ln 2 \sum_{n=1}^{N} \alpha_n w_n} \tag{13}$$

Furthermore, letting

$$\beta_{n+1} = \frac{\left[\prod_{k=1}^{k_1} h_k\right]^{\frac{\alpha_{n+1}}{K_n}}}{\left[\prod_{k=k_n+1}^{k_{n+1}} h_k\right]^{\frac{1}{K_n}}} \quad \text{for } n = 1, 2, 3, ...N-1,$$

$$\gamma_{n+1} = \frac{K_1 \alpha_{n+1}}{K_n} \quad \text{for } n = 1, 2, 3, ...N-1,$$

the relationship between the various water levels dictated by (7) can be succinctly written as:

$$w_n = \beta_n w_1^{\gamma_n} \quad \text{for } n = 2, 3, 4, ...N. \tag{14}$$

After defining $\beta_1 = \gamma_1 = 1$, (13) can be re-written using w_1 alone as:

$$F(w_1) = f(w_1, \beta_2 w_1^{\gamma_2}, \beta_3 w_1^{\gamma_3}, ...\beta_N w_1^{\gamma_N}) - \frac{\sum_{n=1}^{N} \alpha_n}{(\ln 2) \sum_{n=1}^{N} \alpha_n \beta_n w_1^{\gamma_n}} = 0 \tag{15}$$

The fact that (5) is a concave optimization problem implies that the above non-linear equation $F(w_1) = 0$ has a unique solution. Nevertheless, it is possible to prove the existence and uniqueness of the solution independently. An outline is as follows: Notice that $F(w_1)$ is continuous when $w_1 > 0$. It can be shown $\lim_{w_1 \to 0^+} F(w_1) = -\infty$ and $\lim_{w_1 \to \infty} F(w_1) = +\infty$. Hence, by Intermediate Value Theorem, $F(w_1) = 0$ has at least one positive solution. Suppose that $F(w_1) = 0$ has two positive solutions. Since $F(w)$ is differentiable for $w_1 > 0$, by Rolle's Theorem, $F'(w_1) = 0$ for some $w_1 > 0$. However, from (15) it can be shown that $F'(w_1) > 0$ for all $w_1 > 0$.

After finding the optimum water level w_1^* for User-1 by solving (15) for w_1, optimum water levels w_n^* for the other users can be calculated using (14). From the water levels, the power levels p_k^* that maximize the total EE can be calculated using (12). This will form a legitimate solution, only if the power levels p_k^* satisfies the inequality in (8). If the p_k^* values obtained do not satisfy (8), we go to case II.

Case II: $\quad tP_T - \sum_{k=1}^{K} y_k = 0$

This is the case where the solution lies on the plane $\sum_{k=1}^{K} p_k = P_T$. This makes the denominator of the objective function a constant and hence, our optimization problem reduces to the following.

$$\max_{\mathbf{P}} \sum_k \log_2 (1 + h_k p_k)$$

subject to

$$\sum_k p_k = P_T$$

This is the familiar OFDMA throughput maximization problem, for which we know the solution:

$$p_k = p_k^{**} = \left[w^{**} - \frac{1}{h_k}\right]^+, \text{ where } w^{**} = \frac{P_T}{K} + \frac{1}{K}\sum_k \frac{1}{h_k} \tag{16}$$

The results of the analysis are now summarized into a power allocation procedure.

Power Allocation Procedure

1) Solve (15) and obtain User-1's water level w_1^*, and then w_n^* and p_k^* from (14) and (12).
2) If $\sum_{k=1}^{K} p_k^* \le P_T$ then $p_k = p_k^*$ for all k.

   Else $p_k = p_k^{**}$ for all k, where p_k^{**} is given by (16).

C. Convergence and Complexity

We will use the well known result that sorting an array of n elements takes, at worst, a time in the order of $n \log_2 n$. Our subchannel assignment protocol starts by sorting the users according to their rate requirements, and then sorting each row and column of the channel matrix according to the gain. This would take a time in the order of $N \log_2 N + NK \log_2 K + KN \log_2 N$. In the worst case scenario, we will need K/N iterations of the outer loop to assign all subchannels. This shows that the algorithm for the subchannel assignment protocol will converge and the complexity is $NK \log_2 K + (K+1)N \log_2 N + K/N$. Since $K >> N$, this reduces to $O(NK \log_2 K)$.

The problem with the convergence of the algorithms in the literature usually comes from the power allocation part [6]. Our power allocation procedure only need to solve a single non-linear equation, regardless of the number of users in the system. In other words, it has constant time complexity and there are extremely efficient solvers that accomplish this in few iterations [9]. Thus, the resource allocation algorithm converges and has complexity $O(NK \log_2 K)$. It should be noted that the algorithm in [6] uses minimum rate constraints as opposed to our proportional rates, and that algorithm also has the same complexity as ours. The algorithm in [6] is based on bi-level optimization, uses an approximate estimate of the EE, and more importantly, the convergence of that algorithm is not proven.

IV. Numerical Results

The details of the simulation parameters are given in Fig. 1. The power solutions obtained from these simulations fell well within the respective maximum total power constraints. In other words, the solutions came from Case-I of the last section. For one realization of the first scenario, the solution of the equation in (15) was found to be $w_1^* = 2.5$ Milli Watts. This produces a maximum system EE of 267.5 bits/Joule/Hz. Note that all the EE values reported here are the system EE over all the subchannels of a particular scenario per unit bandwidth of each subchannel. The total transmission rate of each user over all the subchannels allocated to it and the total power expenditure of each user at the maximum EE are shown in Table I. The transmission rates as ratios of User-1's rate show the proportional rate demands being satisfied.

Users were placed randomly within a cell radius R.

Raleigh fading and log-normal shadowing were superimposed on the channel gains obtained from the distance to the transmitter using a path loss coefficient of 4.

SCENARIO - 1:
N = 4 Users   K=32 subchannels   R = 400 m
Proportional rate requirement: $r_1 : r_2 : r_3 : r_4 = 1.0 : 0.8 : 0.6 : 0.4$

SCENARIO - 2:
N = 6 Users   K=60 subchannels   R varied from 300 m to 1000 m
No rate requirements.

SCENARIO - 3:
R = 400 m   K = 32, 64, 128
N=4 $r_1 : r_2 : r_3 : r_4 = 1.0 : 0.8 : 0.6 : 0.4$
N=8 $r_1 : r_2 : r_3 : r_4 : r_5 : r_6 : r_7 : r_8 = 1 : 1 : 0.8 : 0.8 : 0.6 : 0.6 : 0.4 : 0.4$
N=12 $r_1 : r_2 : r_3 : r_4 : r_5 : r_6 : r_7 : r_8 : r_9 : r_{10} : r_{11} : r_{12} = 1 : 1 : 1 : 0.8 : 0.8 : 0.8 : 0.6 : 0.6 : 0.6 : 0.4 : 0.4 : 0.4$
N=16 $r_1 : r_2 : r_3 : r_4 : r_5 : r_6 : r_7 : r_8 : r_9 : r_{10} : r_{11} : r_{12} : r_{13} : r_{14} : r_{15} : r_{16} = 1 : 1 : 1 : 1 : 0.8 : 0.8 : 0.8 : 0.8 : 0.6 : 0.6 : 0.6 : 0.6 : 0.4 : 0.4 : 0.4 : 0{:}4$

Fig. 1. Simulation details.

TABLE I
SCENARIO 1: POWER LEVELS AND TRANSMISSION RATES AT THE MAXIMUM EE.

User Number	1	2	3	4
Power level (Milli Watts)	15.3	7.3	6.9	3.7
Transmission rate (bits/s/Hz)	15.68	12.54	9.41	6.27
Transmission rate / User-1's rate	1.00	0.80	0.60	0.40

Since the bi-level optimization based algorithm in [6] and our algorithm use different types of rate constraints, a direct comparison is not possible. We modified these two algorithms to ignore the rate demands and used the second scenario in Fig. 1 to compare them. Fig. 2 shows our algorithm out-performing the one in [6]. For each cell radius, the EE values shown are the averages from 50 random placements of the users.

Scenario-3 is used to investigate the variation of the system EE with the number of users when the number of subchannels is fixed at three different values. The results are shown in Fig. 3. Each point in the graph is obtained by averaging the EE values from 50 random placements. There is a slight decrease in EE with increasing number of users. This can be explained by the decrease in the effectiveness of the channel assignment protocol with the decrease in the subchannel to user ratio. The EE decreases because the channel assignment protocol has lesser and lesser channels per user to work with.

V. CONCLUSION

A two-step solution to the problem of finding the power and frequency allocation that maximizes the bits/Joule/Hz energy efficiency of the OFDMA based downlink transmissions from a base station with proportional rate constraints was presented. The complexity of the algorithm proposed was the same as that of the best algorithm in the literature for the same problem but with minimum constraints. While the algorithm in the literature is not guaranteed to converge, the algorithm proposed in this paper is proven to converge.

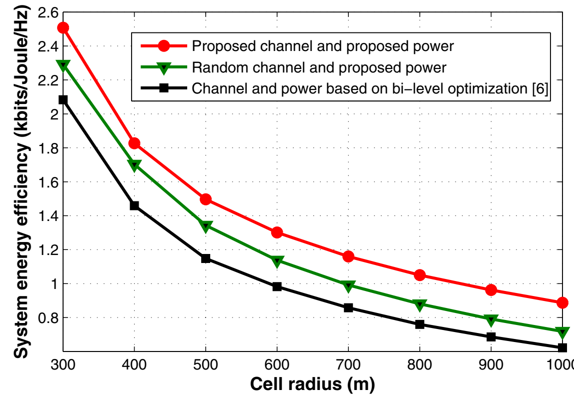

Fig. 2. Scenario-2: Variation of the maximum EE with cell radius.

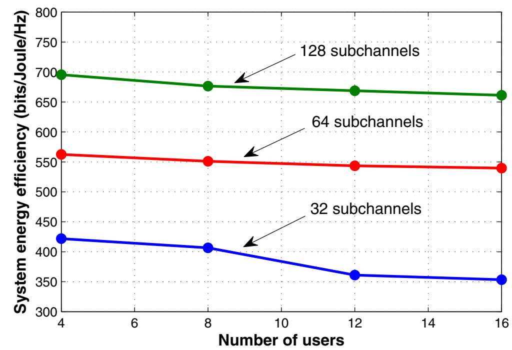

Fig. 3. Scenario-3: Variation of the maximum EE with number of users.

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