Downlink Power Control in Two-Tier Cellular OFDMA Networks Under Uncertainties: A Robust Stackelberg Game

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***Abstract*—We consider the problem of robust downlink power control in orthogonal frequency-division multiple access (OFDMA)-based heterogeneous wireless networks (HetNets) composed of macrocells and underlaying small cells. A non-cooperative setting is assumed where the macro base stations (MBSs) and small cell base stations (SBSs) compete with each other to maximize their own capacities considering imperfect channel state information. A robust Stackelberg game (RSG) is formulated to model this hierarchical competition where the MBSs and SBSs act as the leaders and the followers, respectively. The formulated RSG can be expressed as an equilibrium program with equilibrium constraints (EPEC). A comprehensive study of this RSG is provided considering various power constraints (e.g., total and spectral mask), various interference constraints (e.g., individual and global), and different uncertainty models (e.g., column-wise and ellipsoidal). We show how the different constraints and uncertainty models change the property of the game (e.g., Nash equilibrium problem (NEP) or generalized Nash equilibrium problem (GNEP)) and accordingly impact the choice of analysis method (e.g., game theory or variational inequality (VI)), solution (e.g,. closed-form or numerical), and the design of algorithms and their distributive properties (e.g., totally distributed, semi-distributed, and centralized). A robust Stackelberg equilibrium (RSE) is considered to be the solution and its existence and uniqueness are investigated. Also, algorithms are proposed to arrive at the RSE. Numerical results show the effectiveness of robust solutions in an imperfect information environment.**

***Index Terms*—Small cell networks, power control, uncertainty, robust Stackelberg game, worst-case analysis, equilibrium program with equilibrium constraints, generalized Nash equilibrium problem, variational inequality.**

I. Introduction

FUTURE wireless networks (e.g., 5G) are expected to be highly heterogeneous composed of macrocells and a large number of small cells with small coverage area and low transmission power. For efficient frequency reuse and spectrum sharing, the co-channel deployment has been regarded as a practical solution where both macrocells and small cells can utilize all available spectrum (i.e., frequency reuse factor of one). With such co-channel deployment, the spectrum-wide cross-tier and co-tier interferences could severely limit the system performance. Accordingly, interference mitigation is of vital importance for which subchannel assignment and power control-based resource allocation has been shown to be a feasible solution for OFDMA-based heterogeneous wireless networks (HetNets) [1]. In this paper, we address the power control problem for downlink transmission in two-tier HetNets.

In HetNets, the base stations (BSs) could be deployed by different parties (e.g., traditional service providers, emerging secondary service providers, and individual users) with different interests. Centralized resource allocation schemes requiring coordination among all BSs may not be practical in HetNets due to the large amount of small cell base stations (SBSs) deployed, privacy protection by the owners, and security issues. Accordingly, we consider a non-cooperative setting where each BS is of self-interest and performs resource allocation independently and distributively aiming to maximize its own capacity. Also, a hierarchical structure of the competition among the macrocell base stations (MBSs) and SBSs is considered where the MBSs are assumed to have privileges over SBSs in making decisions, and the mathematical modeling fits naturally into the framework of a Stackelberg game.

Game theory-based power control has been widely studied in the literature. Many existing works [2]–[7] have formulated the non-cooperative power control problem in different wireless systems as Nash equilibrium problems (NEPs), and have proposed various water-filling algorithms (e.g., iterative [2], [3], simultaneous [4], and asynchronous [5]) to achieve the Nash equilibrium (NE) in a distributed manner. Compared with the centralized power allocation schemes, these game theory-based distributed algorithms require less or even no information exchange.

Stackelberg game has also been applied in several works [8]–[10] to explicitly capture the hierarchical competition in two-tier HetNets with different design purposes. Specifically, in [9], [10], pricing-based single-leader multi-follower Stackelberg games are formulated, where the MBS as the single leader prices the interference from the femtocell users and the objective of the MBS is to maximize its monetary revenue. A rate maximization Stackelberg game is formulated in [8]. However, all these works are based on a common assumption of perfect information, where all the parameters involved in the objective function and constraints of the individual

Manuscript received May 20, 2014; revised September 20, 2014 and November 25, 2014; accepted December 7, 2014. Date of publication December 18, 2014; date of current version February 12, 2015. This work was supported by a Strategic Project Grant (STPGP 430285) from the Natural Sciences and Engineering Research Council of Canada (NSERC). The associate editor coordinating the review of this paper and approving it for publication was P. Popovski.

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Digital Object Identifier 10.1109/TCOMM.2014.2382095

optimization problem can be precisely obtained. Due to the dynamic and random nature of wireless environment and limited amount of coordination in HetNets, information uncertainties naturally appear and the assumption of perfect information may not be practical. Accordingly, when applying perfect information based resource allocation schemes in a practical system with parameter uncertainties, possible performance degradation could be experienced. For example, interference constraints are usually imposed to protect high priority users. However, these interference constraints could be violated due to the uncertainty in channel state information (CSI) in a system using a perfect information based scheme. Therefore, a robust design of resource allocation is required which takes the imperfect information into account.

To explicitly model the parameter uncertainty, the actual value of the parameter is usually represented by the sum of an estimated nominal value and a perturbation term which is also regarded as the uncertainty part. Similar to that in optimization theory, two approaches are used in game theory literature to deal with the information uncertainties: a Bayesian approach considering average payoff optimization which leads to the development of Bayesian games, and a worst-case approach based on robust optimization [11] for worst-case payoff optimization which leads to the development of robust games [12].

The Bayesian approach considers the uncertainty term as a random variable the probability distribution of which is assumed to be known, and the performance is guaranteed on an average basis. For the cases where the probability distribution is not available, the distribution-free worst-case approach is a better option which only requires information about the bound of uncertain parameters. Specifically, with the worst-case approach, the uncertainty term is assumed to be bounded within an uncertainty region, and certain level of performance can be guaranteed for every realization in the uncertainty set which can prevent undesirable performance fluctuations.

In this work, we consider the uncertainty in CSI of interfering channels and follow the worst-case approach. Several existing works [13], [15], [16], [28] have also considered the worst-case robust formulation of non-cooperative power control games in different settings. Specifically, a robust non-cooperative power control game is formulated in [13] for modeling the competitive rate maximization considering bounded channel uncertainty in a frequency-selective Gaussian channel. However, no interference constraint is considered. Similarly, interference constraint is not considered in the power control game of [28] which is an example of the robust additively coupled games. Game formulations for rate maximization in cognitive radio networks are presented in [15], [16]. However, only individual interference constraints are considered and there is no uncertainty in the payoff function of [16].

More importantly, all these works only consider the simultaneous play formulations while the hierarchical competition among the players is largely ignored. Attempts to obtain robust solutions for Stackelberg games considering uncertain observations can be found in [17], [27], [28]. In this work, *we consider a comprehensive robust formulation of the Stackelberg game for power control in two-tier OFDMA cellular networks, which has not been investigated in the existing literature*.

Specifically, we formulate the downlink power control problem in a two-tier HetNet with imperfect CSI as a distribution-free multi-leader multi-follower robust Stackelberg game (RSG), where the MBSs are considered to be the leaders while the SBSs are the followers. The RSG is composed of a leader sub-game and a follower sub-game and accordingly can be expressed as an equilibrium program with equilibrium constraints (EPEC) for which we present a comprehensive investigation by considering various power and interference constraints, and different types of uncertainty models. We show how the different considerations of the constraints and uncertainty models change the property of the game, and accordingly impact the choice of the analysis method, the form of solution, and the design of algorithms and their distributive properties.

Robust Stackelberg equilibrium (RSE) composed of a robust Nash equilibrium (RNE) for the follower sub-game and a RNE for the leader sub-game is considered to be the solution of the RSG, and the existence and uniqueness of the solution are investigated. Closed-form exact solutions are obtained for a single-leader multiple-follower formulation with column-wise uncertainty model for both sparse and dense BS deployment scenarios. When ellipsoidal uncertainty model is considered, a closed-form solution for the follower sub-game is possible, while a closed-form solution of the RSE is not available. Instead, algorithms are proposed for reaching the RSE. Numerical results show the effectiveness of the RSE in terms of performance improvement in imperfect CSI environments.

The rest of the paper is organized as follows. Section II describes the system model, assumptions, and presents the robust Stackelberg game framework. In Section III, given the power allocation of MBSs, the follower sub-game is analyzed considering different power and interference constraints. The leader sub-game is then analyzed in Section IV considering different formulations for the follower sub-game. Numerical results are presented in Section V. Section VI concludes the paper. Brief introductions to Stackelberg game, variational inequality, and robust optimization are provided in Appendices A, B, and C, respectively.

II. System Model, Assumptions, and Game Formulation

A. System Model and Assumptions

We consider downlink transmissions in an OFDMA-based two-tier celluar network consisting of a set $\mathcal{M} = \{1, 2, \ldots, M\}$ of macrocells and a set $\mathcal{K} = \{1, 2, \ldots, K\}$ of small cells sharing the same set $\mathcal{N} = \{1, 2, \ldots, N\}$ of orthogonal subchannels. Also, we assume that only one user is served by each MBS/SBS in each subchannel. Denote by $g_{lm}(n)$ the co-tier channel power gain between the MBS l and the user of MBS m on subchannel n. The cross-tier channel power gain between the MBS m and user of SBS k on subchannel n is denoted by $h_{mk}(n)$. Similarly, we can define the co-tier channel power gain between SBSs as $g_{jk}(n)$ and the cross-tier channel power gain between SBS k and user of MBS m as $h_{km}(n)$. We assume that the subchannels experience slow and flat fading.

Imperfect CSI is considered which brings uncertainty into the system. Specifically, the actual value of the channel gain is expressed as the sum of an estimated nominal value and an uncertainty term as follows:

$$g_{lm}(n) = \bar{g}_{lm}(n) + \triangle g_{lm}(n), \tag{1}$$

where $\bar{g}_{lm}(n)$ is the nominal value and $\triangle g_{lm}(n)$ is the uncertainty term. Several models have been applied in the lietrature to describe the parameter uncertainties (e.g., general polyhedron, D-norm, ellipsoidal, and column-wise [18]) among which the column-wise and ellipsoidal models are most widely used due to their analytical tractability.

With column-wise model, the channel uncertainty is modeled by

$$g_{lm}(n) = \left\{\bar{g}_{lm}(n) + \triangle g_{lm}(n) : |\triangle g_{lm}(n)| \le \varepsilon_{lm}(n)\right\},$$

where $\varepsilon_{lm}(n)$ is the column-wise uncertainty bound. With ellipsoidal model, the channel uncertainty is described by

$$g_{lm}(n) = \left\{\bar{g}_{lm}(n) + \triangle g_{lm}(n) : \|\triangle g_{lm}(n)\|_{\mathbf{w}_{lm}(n)} \le \varepsilon_m(n)\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector and $\mathbf{w}_{lm}(n)$ is a vector of positive weights. Similarly, the uncertainty models for $g_{jk}(n)$, $h_{mk}(n)$, and $h_{km}(n)$ can be obtained.

The transmit powers of MBS m and SBS k are denoted by $\mathbf{p}_m = [p_m(1), \ldots, p_m(n), \ldots, p_M(N)]$ and $\mathbf{p}_k = [p_k(1), \ldots, p_k(n), \ldots, p_K(N)]$, respectively, where $p_m(n) \ge 0$ and $p_k(n) \ge 0$ are the transmit powers of MBS m and SBS k on subchannel n, respectively. For each MBS and SBS, the total transmit power is limited by

$$\sum_{n=1}^{N} p_m(n) \le P_m^{\text{sum}}, \quad \sum_{n=1}^{N} p_k(n) \le P_k^{\text{sum}}, \tag{2}$$

where P_m^{sum} and P_k^{sum} are the corresponding power budgets. Besides the total power constraint, a spectral mask constraint could also be imposed to limit the maximum transmit power allocated to each subchannel as $p_m(n) \le P_m^{\max}(n)$ and $p_k(n) \le P_k^{\max}(n)$.

To protect macro users from excessive interference from SBSs, two types of interference constraints can be imposed at the SBS side: an individual interference constraint

$$p_k(n) h_{km}(n) \le I_{km}^{\max}(n), \quad \forall k, m, n, \tag{3}$$

which limits the interference generated by each SBS and a global interference constraint

$$\sum_{k=1}^{K} p_k(n) h_{km}(n) \le I_m^{\max}(n), \quad \forall m, n, \tag{4}$$

which limits the aggregate interference generated by all SBSs.

The MBSs and SBSs are considered to be of self-interest aiming to maximize their own achievable rate through the allocation of transmit power over the N subchannels subject to certain power and interference constraints. Given the power allocations of all MBSs and SBSs, the interference plus noise experienced by users of MBS m and SBS k on channel n can be expressed as $v_m(n) = \sigma_m^2(n) + \sum_{m \ne l} g_{lm}(n) p_l(n) + \sum_{k=1}^{K} h_{km}(n) p_k(n)$, and $v_k(n) = \sigma_k^2(n) + \sum_{m=1}^{M} h_{mk} p_m(n) + \sum_{j \ne k} g_{jk}(n) p_j(n)$, respectively. Accordingly, the maximum downlink transmission rate for MBS m and SBS k can be obtained as follows:

$$\mathcal{R}_m(\mathbf{p}_m, \mathbf{p}_{-m}, \mathbf{p}^{\text{low}}) = \sum_{n=1}^{N} \log\left(1 + p_m(n)/v_m(n)\right),$$

$$\mathcal{R}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}}) = \sum_{n=1}^{N} \log\left(1 + p_k(n)/v_k(n)\right),$$

where $\mathbf{p}_{-m}$ and $\mathbf{p}_{-k}$ represent the power allocations of all MBSs expect the mth MBS and power allocations of all SBSs expect the kth SBS, respectively, $\mathbf{p}^{\text{low}}$ and $\mathbf{p}^{\text{up}}$ represent the vectors of power allocations of all SBSs and MBSs, respectively. Note that the gains of the interfering channels are normalized to make all direct channel gains to be one (i.e., $g_{mm}(n) = 1$ and $g_{kk}(n) = 1$).

B. Robust Stackelberg Game Formulation

A robust Stackelberg game is formulated to capture the hierarchical competition among the MBSs and SBSs. Also, the competitive interactions exist within the multiple leaders and the multiple followers. In this case, a robust non-cooperative leader sub-game for the leaders and a robust non-cooperative follower sub-game for the followers are formulated. Both of them together constitute the robust Stackelberg game. Specifically, the structure of the robust Stackelberg game is described as follows:

- *Players*: The MBSs and SBSs are the players of the robust Stackelberg game. Also, the MBSs are the players (as the leaders) of the leader sub-game and the SBSs are the players (as the followers) of the follower sub-game.
- *Strategy*: The strategy of each player (both the leaders and the followers) is the selection of power allocation over the N available subchannels.
- *Strategy space*: Denote by $\mathcal{P}_m$ and $\mathcal{P}_k$ the sets of admissible power allocation of MBS m and SBS k, respectively. The strategy spaces of the leaders and the followers are then given by $\mathcal{P}^{\text{up}} = \prod_{m \in \mathcal{M}} \mathcal{P}_m$ and $\mathcal{P}^{\text{low}} = \prod_{k \in \mathcal{K}} \mathcal{P}_k$, respectively. The strategy space of the entire game is given by the Cartesian product $\mathcal{P}^{\text{up}} \times \mathcal{P}^{\text{low}}$. Note that $\mathcal{P}_m$ and $\mathcal{P}_k$ vary with different considerations of the constraints.
- *Uncertainty models:* Both column-wise and ellipsoidal uncertainty models are considered.
- *Robustness*: For robustness analysis, we consider the worst-case.
- *Payoff*: The payoffs of the MBSs and SBSs are defined as $\pi_m = \mathcal{R}_m(\mathbf{p}_m, \mathbf{p}_{-m}, \mathbf{p}^{\text{low}})$ and $\pi_k = \mathcal{R}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}})$, respectively.

Robust Stackelberg equilibrium (RSE) is considered to be the solution of the game which is constituted by a robust Nash equilibrium (RNE) of the leader sub-game $\mathbf{p}^{\text{up}*}$ and a RNE of the follower sub-game $\mathbf{p}^{\text{low}*}(\mathbf{p}^{\text{up}*})$, and is represented by the tuple $\{\mathbf{p}^{\text{up}*}, \mathbf{p}^{\text{low}*}(\mathbf{p}^{\text{up}*})\}$.

In the following, the game is analyzed considering four types of constraints. For each type of constraint, we further consider two types of uncertainty models.

III. Robust Follower Sub-Game

Backward induction is commonly used to analyze a Stackelberg game. Therefore, we start analyzing the robust Stackelberg game by firstly investigating the robust follower sub-game given the strategies of the leaders. Various power and interference constraints and two uncertainty models are considered.

A. With Only Total Power Constraint

We start analyzing the follower sub-game considering only a total maximum transmit power constraint for each SBS. In this case, the admissible strategy set of SBS k is

$$\mathcal{P}_k = \left\{ \mathbf{p}_k \in \mathbb{R}^N : \sum_{n=1}^{N} p_k(n) \le P_k, p_k(n) \ge 0 \right\}, \forall k \in \mathcal{K}.$$

The nominal formulation of the follower sub-game is described as follows. Given the power allocations $\mathbf{p}^{\text{up}}$ of the leaders, each follower k aims to maximize its own achievable downlink transmission rate as

$$\begin{array}{ll} \max\limits_{\mathbf{p}_k} & \mathcal{R}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}}) \\ \text{s.t.} & \mathbf{p}_k \in \mathcal{P}_k. \end{array} \tag{5}$$

The robust counterpart of the nominal game can be formulated by considering uncertainties in the channel state information. Specifically, each follower k aims to robustly maximize its downlink rate subject to CSI uncertainty which is a max-min problem described as follows:

$$\max_{\mathbf{p}_k} \min_{\mathbf{g}_k, \mathbf{h}_k} \mathcal{R}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}}) \tag{6}$$

$$\text{s.t.} \quad \mathbf{p}_k \in \mathcal{P}_k,\ \mathbf{g}_k \in \mathcal{G}_k,\ \mathbf{h}_k \in \mathcal{H}_k, \tag{7}$$

where $\mathbf{g}_k = \{g_{jk}\}_{j \ne k, j \in \mathcal{K}}$ and $\mathbf{h}_k = \{h_{mk}\}_{m \in \mathcal{M}}$. $\mathcal{G}_k$ and $\mathcal{H}_k$ are the information uncertainty sets of player k. Note that since there is only power constraint, the uncertainties appear only in the payoff function.

1) With Column-Wise Uncertainty Model: In this case, the uncertainty sets for follower k are modeled as

$$\mathcal{G}_k = \left\{ \bar{g}_{jk}(n) + \triangle g_{jk}(n),\ \left|\triangle g_{jk}(n)\right| \le \epsilon_{jk}(n) \right\}_{j \ne k, j \in \mathcal{K}},$$

$$\mathcal{H}_k = \left\{ \bar{h}_{mk}(n) + \triangle h_{mk}(n),\ \left|\triangle h_{mk}(n)\right| \le \varepsilon_{mk}(n) \right\}_{m \in \mathcal{M}}.$$

With this uncertainty model, the worst-case analysis of the problem in (6) becomes

$$\max_{\mathbf{p}_k \in \mathcal{P}_k} \sum_{n=1}^{N} \log\left(1 + p_k(n)/\tilde{v}_k(n)\right) \tag{8}$$

where

$$\tilde{v}_k(n) = \sigma_k^2 + \sum_{m=1}^{M} \left(\bar{h}_{mk}(n) + \varepsilon_{mk}(n)\right) p_m(n) + \sum_{j \ne k} \left(\bar{g}_{jk}(n) + \epsilon_{jk}(n)\right) p_j(n). \tag{9}$$

In this case, the analysis of the robust game could immediately follow its nominal counterpart with the replacement of $g_{jk}(n)$ and $h_{mk}(n)$ with $\bar{g}_{jk}(n) + \epsilon_{jk}(n)$ and $\bar{h}_{mk}(n) + \varepsilon_{mk}(n)$, respectively. When the strategies of all other followers $\mathbf{p}_{-k}$ are fixed, the optimization problem for each follower is convex [29] and the solution is the water-filling solution which can be obtained from the KKT condition as follows:

$$p_k(n) = \begin{cases} \mu_k - \tilde{v}_k(n), & \mu_k > \tilde{v}_k(n), \\ 0, & \mu_k \le \tilde{v}_k(n), \end{cases} \tag{10}$$

or simply $p_k(n) = \mathbf{WF}_k(\mathbf{p}_{-k}, \mathbf{p}^{\text{up}}) = [\mu_k - \tilde{v}_k(n)]^+$, where $[x]^+ \triangleq \max(x, 0)$, $\frac{1}{\mu_k}$ is the positive Lagrange multiplier, and μ_k is regarded as the water level. Note that the total power constraint in (2) must be satisfied with equality for optimality. Accordingly, the problem for each follower becomes finding a water level such that the total power constraint is satisfied, i.e., $\sum_{n=1}^{N} p_k(n) = \sum_{n=1}^{N} [\mu_k - \tilde{v}_k(n)]^+ = P_k^{\text{sum}}$.

Both exact and iterative algorithms can be used to find the water level. A generalization of the algorithms for finding the water level can be found in [19]. Note that in this formulation the water level of a follower is the same for all subchannels. For the other formulations, as we will see later in this paper, these exist cases where variable water levels exist for different subchannels.

The existence of RNE can be easily proved by showing that the formulated follower sub-game is a concave game in which the payoff of each player is continuous and concave and the admissible strategies are compact and convex. While the sufficient condition for the uniqueness of RNE can also be obtained by following the uniqueness analysis of the nominal counterpart. This has been well studied in [2], [4], [5], mostly through the use of fixed point theory (the uniqueness of a fixed point of the best response function can be guaranteed if the best response function is a contraction in certain norm [7]), and therefore, is omitted here.

To arrive at the equilibrium power allocations, the water-filling algorithms in the literature (e.g., iterative, simultaneous, and asynchronous) can be used. However, it is worth noting that the introduction of uncertainty changes the distributive property of the algorithms. Specifically, in the nominal game, the interference from the MBSs and other SBSs can be measured by each player without any information exchange. However, in the algorithms based on the robust formulation, exchange of information about the power allocation strategies of other players is required.

2) With Ellipsoidal Uncertainty Model: This model provides more flexibility than the column-wise uncertainty model. The ellipsoidal uncertainty sets for follower k are modeled as

$$\mathcal{G}_k = \left\{ \bar{g}_{jk}(n) + \triangle g_{jk}(n),\ \left\|\triangle g_{jk}(n)\right\|_{\mathbf{w}_{jk}(n)} \le \epsilon_k^2(n) \right\}_{j \ne k, j \in \mathcal{K}},$$

$$\mathcal{H}_k = \left\{ \bar{h}_{mk}(n) + \triangle h_{mk}(n),\ \left\|\triangle h_{mk}(n)\right\|_{\mathbf{w}_{mk}(n)} \le \varepsilon_k^2(n) \right\}_{m \in \mathcal{M}}.$$

With the ellipsoidal uncertainty model, for worst-case analysis, the problem in (6) can be transformed as

$$\max_{\mathbf{p}_k \in \mathcal{P}_k} \hat{\mathcal{R}}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\mathrm{up}}) \tag{11}$$

where

$$\hat{\mathcal{R}}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\mathrm{up}}) = \sum_{n=1}^{N} \log\left(1 + \frac{p_k(n)}{\hat{v}_k(n)}\right), \tag{12}$$

and

$$\hat{v}_k(n) = \sigma_k^2(n) + \sum_{m=1}^{M} \bar{h}_{mk}(n) p_m(n) + \sum_{j \neq k} \bar{g}_{jk}(n) p_j(n) + \epsilon_k(n)\sqrt{\sum_{j\neq k} \frac{|p_j(n)|^2}{w_{jk}(n)}} + \varepsilon_k(n)\sqrt{\sum_{m=1}^{M} \frac{|p_m(n)|^2}{w_{mk}(n)}}. \tag{13}$$

The derivation is omitted for brevity. Given the strategies of the MBSs and all other SBSs, the optimization problem in (11) is still convex. Applying KKT optimality conditions, a closed-form solution for the robust power control can be obtained, which is a water-filling like solution, given as

$$\mathbf{p}_k^* = [\mu_k - \hat{v}_k(n)]^+. \tag{14}$$

The existence of RNE for the formulation with ellipsoidal model can be established in a way similar to that with columnwise model. The sufficient condition for the uniqueness of RNE can be obtained in a way similar to that in [13].

To arrive at the RNE of the lower sub-game, a robust version of the water-filling algorithm can be used with the replacement of the traditional water-filling solution with (14) which is also shown in [13]. Also, the totally distributive property of the water-filling algorithm for the nominal game cannot be preserved for the robust version. In the remaining analysis of the follower sub-game, we will focus on the ellipsoidal uncertainty model.

B. With Total Power and Spectral Mask Constraints

Besides the total power constraint, the spectral mask constraints (i.e., $p_k(n) \le P_k^{\max}(n), \forall n$) could also be introduced. In this case, the admissible strategy set of each follower k is

$$\mathcal{P}_k = \left\{ \mathbf{p}_k \in \mathbb{R}^N : \sum_{n=1}^{N} p_k(n) \le P_k^{\mathrm{sum}}, 0 \le p_k(n) \le P_k^{\max}(n) \right\}.$$

Note that we assume $P_k^{\max}(n) < P_k^{\mathrm{sum}} < \sum_{n=1}^{N} P_k^{\max}(n)$ to avoid a trivial solution.

Adding the spectral mask constraints does not change the convexity of the problem in (6). It is not difficult to obtain a closed-form solution for follower k as $\mathbf{p}_k^* = [\mu_k - \hat{v}_k(n)]_0^{P_k^{\max}(n)}$, where $[x]_a^b$ is the Euclidean projection of x onto the interval $[a, b]$, and $\hat{v}_k(n)$ is shown in (13).

It is worth noting that, the introduction of spectral mask constraints in the follower sub-game does not change the analysis of the follower sub-game itself; however, it will impact the analysis of the leader sub-game which will be shown in Section IV.

C. With Total Power, Spectral Mask, and Individual Interference Constraints

To limit the interference caused by the SBSs to macro users, an individual interference constraint can be imposed to each SBS. In this case, the admissible strategy set can be expressed as $\mathcal{P}_k = \bar{\mathcal{P}}_k \bigcap \tilde{\mathcal{P}}_k$, where

$$\bar{\mathcal{P}}_k = \left\{ \mathbf{p}_k \in \mathbb{R}^N : \sum_{n=1}^{N} p_k(n) \le P_k, 0 \le p_k(n) \le P_k^{\max}(n) \right\},$$

is the set of feasible power allocations satisfying the power constraints and

$$\tilde{\mathcal{P}}_k = \left\{ \mathbf{p}_k \in \mathbb{R}^N : p_k(n) h_{km}(n) \le I_{km}^{\max}(n) \right\},$$

is the set of admissible power allocations satisfying the individual interference constraints.

Note that the channel state information is required to impose the interference constraint. Accordingly, uncertainties could also appear in the constraints of the robust rate maximization problem. In this work, we consider the uncertainties in both payoff function and constraints. Accordingly, besides the uncertainty sets $\mathcal{G}_k$ and $\mathcal{H}_k$, we also consider uncertainty in the individual interference constraint which is deterministically modeled as

$$\mathcal{H}_{km} = \left\{ \bar{h}_{km}(n) + \triangle h_{km}(n), \|\triangle h_{km}(n)\|_{\mathbf{w}_{km}} \le \varepsilon_m^2(n) \right\}.$$

And the robust rate maximization of each player becomes

$$\begin{aligned} &\max_{\mathbf{p}_k} \min_{\mathbf{g}_k, \mathbf{h}_k} \mathcal{R}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\mathrm{up}}) \\ \text{s.t.} \quad &\max_{h_{km}(n)} p_k(n) h_{km}(n) \le I_{km}^{\max}(n), \quad \forall m, n, \\ &\mathbf{p}_k \in \mathcal{P}_k, \mathbf{g}_k \in \mathcal{G}_k, \mathbf{h}_k \in \mathcal{H}_k, h_{km}(n) \in \mathcal{H}_{km}. \end{aligned} \tag{15}$$

Following the worst-case approach, the robust individual interference constraint can be restated as

$$p_k(n) \left(\bar{h}_{km}(n) + \varepsilon_m(n) / \sqrt{w_{km}(n)} \right) \le I_{km}^{\max}(n).$$

Then it is straightforward to combine the spectral mask constraint and the robust individual interference constraint as follows:

$$p_k(n) \le \min\left\{ P_k^{\max}(n), \frac{I_{km}^{\max}(n)}{\bar{h}_{km}(n) + \varepsilon_m(n)/\sqrt{w_{km}(n)}} \right\}.$$

Accordingly, the following analysis is the same as the case with only total power and spectral mask constraint, and therefore, omitted here for brevity.

Remark: Note that these constraints cannot be simply combined if average power and/or average interference constraints are considered as shown in [6].

D. With Total Power, Spectral Mask, and Aggregate Interference Constraints

The individual interference constraint could be too conservative which may limit the performance of the SBSs. To be more flexible, a global aggregate interference constraint can be imposed to limit the aggregate interference caused by all SBSs. In this case, the admissible strategy set for each SBS can be expressed as $\mathcal{P}_k = \bar{\mathcal{P}}_k \bigcap \hat{\mathcal{P}}_k$, where

$$\hat{\mathcal{P}}_k = \left\{ \mathbf{p}_k \in \mathbb{R}^N : \sum_{k=1}^{K} p_k(n) h_{km}^n \le I_m^{\max}(n) \right\} \tag{16}$$

is the set of feasible power allocations satisfying the global interference constraint.

And the robust rate maximization for each follower becomes

$$\begin{aligned} &\max_{\mathbf{p}_k} \min_{\mathbf{g}_k, \mathbf{h}_k} \mathcal{R}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}}) \\ &\text{s.t. } \max_{h_{km}(n)} \sum_{k=1}^{K} p_k(n) h_{km}^n \le I_m^{\max}(n), \quad \forall m, n, \\ &\mathbf{p}_k \in \mathcal{P}_k,\, \mathbf{g}_k \in \mathcal{G}_k,\, \mathbf{h}_k \in \mathcal{H}_k,\, h_{km}(n) \in \mathcal{H}_{km}. \end{aligned} \tag{17}$$

Different from the individual interference constraint, the global interference constraint brings the coupling among the admissible power allocations of all followers. That is, the admissible strategy set of follower k is not independent and fixed but depends on the strategies chosen by all other followers, i.e., $\hat{\mathcal{P}}_k = \hat{\mathcal{P}}_k(\mathbf{p}_{-k})$. In this case, the previous Nash equilibrium problems (in which the interactions of players only appear in the payoff functions) become a generalized Nash equilibrium problem (GNEP) [20] (in which the interactions of players appear in both payoffs and admissible sets).

Remark: For the formulation with column-wise uncertainty model, a water-filling like solution can be obtained by applying the KKT conditions as follows (the derivation is omitted here for brevity):

$$p_k(n) = \left[\frac{1}{\mu_k + \sum_{m=1}^{M} \lambda_m(n) \left(\bar{h}_{km}(n) + \varepsilon_{km}(n) \right)} - \tilde{v}_k(n) \right]_0^{P_k^{\max}(n)},$$

where $\tilde{v}_k(n)$ is defined in (9). However, the analysis of this GNEP is still very difficult due to the coupled constraint of admissible sets. For the formulation with ellipsoidal uncertainty model, even a closed-form solution is not available.

In general, a GNEP problem is not tractable due to the variability of the admissible sets. A common method to analyze a GNEP is to reformulate it as an equivalent better known problem which could serve as the basis for theoretical analysis (e.g., existence and uniqueness) and algorithm design. Following this method, we first show that the formulated follower sub-game (17) as a GNEP is equivalent to a quasi-variational inequality (QVI) problem [24] the definition of which is given as follows:

Definition 1: Given a closed and convex set $\mathbf{X}(\mathbf{x})$ and a mapping $\mathbf{F} : \mathbf{X}(\mathbf{x}) \mapsto \mathbb{R}^N$, the quasi-variational inequality problem $QVI(\mathbf{X}(\mathbf{x}), \mathbf{F})$ is to find a vector $\mathbf{x}^*$ such that

$$(\mathbf{x} - \mathbf{x}^*)^{\text{T}} \mathbf{F}(\mathbf{x}^*) \ge 0, \quad \forall \mathbf{x} \in \mathbf{X}(\mathbf{x}^*).$$

With the definition of a QVI problem, the equivalence of the formulated GNEP and QVI is shown in the following lemma.

Lemma 3.1: The robust follower sub-game with global robust interference constraint is equivalent to $QVI(\mathbf{X}(\mathbf{p}^{\text{low}}), \mathbf{F}(\mathbf{p}^{\text{low}}))$, where

$$\mathbf{X}(\mathbf{p}^{\text{low}}) \triangleq \prod_{k=1}^{K} \bar{\mathcal{P}}_k \bigcap \hat{\mathcal{P}}_k(\mathbf{p}_{-k}), \text{ and}$$

$$\mathbf{F}(\mathbf{p}^{\text{low}}) \triangleq \left(-\nabla_{\mathbf{p}_k} \hat{\mathcal{R}}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}}) \right)_{k=1}^{K},$$

in which $\hat{\mathcal{R}}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}})$ is defined in (12).

Proof: See Appendix D. ■

The equivalence is established in the sense that a power allocation vector $\mathbf{p}^{\text{low}*}$ is a generalized Nash equilibrium of the follower sub-game if and only if it is a solution of the above QVI problem.

The analysis of the above $QVI(\mathbf{X}(\mathbf{p}^{\text{low}}), \mathbf{F}(\mathbf{p}^{\text{low}}))$ is still challenging due to the lack of developed theory for QVI problems. Fortunately, for a special class of GNEP (i.e., jointly convex GNEP [20]), the equivalent QVI formulation can be reduced to a variational inequality (VI) problem which has been well studied.

Let us consider a GNEP in which the strategy space of each player k is defined as $\mathbf{x}_k \in \mathbf{X}_k(\mathbf{x}_{-k})$. Then the GNEP is a jointly convex GNEP if there exists a closed convex set $\mathbf{X}$ such that for all players, the condition $\mathbf{X}_k(\mathbf{x}_{-k}) = \{\mathbf{x}_k : (\mathbf{x}_k, \mathbf{x}_{-k}) \in \mathbf{X}\}$ holds.

In our case, the jointly convex condition requires that the coupled constraint is common to all players. It can be noticed that the follower sub-game fits into the category of jointly convex GNEP since the coupled constraint $\hat{\mathcal{P}}_k$ is the same for all $k \in \mathcal{K}$ which can therefore be termed as $\hat{\mathcal{P}}$. In this case, the $QVI(\mathbf{X}(\mathbf{p}^{\text{low}}), \mathbf{F}(\mathbf{p}^{\text{low}}))$ can be simplified as a variational inequality problem as $VI(\mathbf{X}, \mathbf{F}(\mathbf{p}^{\text{low}}))$, where $\mathbf{F}(\mathbf{p}^{\text{low}})$ is the same as that in the QVI formulation, and $\mathbf{X} \triangleq \hat{\mathcal{P}} \bigcap \prod_{k=1}^{K} \bar{\mathcal{P}}_k$. Then finding an equilibrium of the game becomes solving the VI problem which is to find a $\mathbf{p}^{\text{low}*} \in \mathbf{X}$ such that

$$(\mathbf{p}^{\text{low}} - \mathbf{p}^{\text{low}*})^{\text{T}} \mathbf{F}(\mathbf{p}^{\text{low}*}) \ge 0, \quad \forall \mathbf{p}^{\text{low}} \in \mathbf{X}. \tag{18}$$

That is, we reformulate the robust follower sub-game as a VI problem. Similar VI reformulations of nominal games can be found in [7], [21].

In the following, based on the finite dimensional variational inequality theory [22], we analyze the existence and uniqueness of the RNE of the follower sub-game, and introduce a centralized algorithm for reaching the RNE. Specifically, the existence of a RNE is shown in the following theorem.

Theorem 3.2: The RNE exists for the follower sub-game with global interference constraint.

Proof: See Appendix E. ■

The analysis of the uniqueness of the solution is more involved (when compared to that of the existence of the solution) which requires further analysis of the mapping $\mathbf{F}(\mathbf{p}^{\text{low}})$. To this end, we first give the definition of strong monotonicity of a mapping.

Definition 2: Given a convex set $\mathbf{X}$, a vector mapping $\mathbf{F}$ is strongly monotone on $\mathbf{X}$ if there exists a constant $\sigma > 0$ such that $(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y}))^{\mathrm{T}}(\mathbf{x}-\mathbf{y}) \geq \sigma\|\mathbf{x}-\mathbf{y}\|^2$ is satisfied for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$.

With the definition of strong monotonicity, we can have the following lemma for the uniqueness of the solution of a variational inequality problem.

Lemma 3.3: The $VI(\mathbf{X},\mathbf{F})$ admits a unique solution if $\mathbf{F}$ is strongly monotone [22].

Basically, the strict monotonicity indicates that the impact of a player's strategy on her own objective function is larger than that from other players' strategies. To investigate the uniqueness of the RNE of the follower game, we extend the analysis for nominal formulations [7], [21] and define a matrix Υ as follows:

$$[\Upsilon]_{kj} \triangleq \begin{cases} 1, & \text{if } k=j, \\ -\max\limits_{1\leq n\leq N}\left\{\bar{g}_{jk}(n)\frac{c}{d}\right\}, & \text{if } k\neq j, \end{cases} \tag{19}$$

where

$$c = \sigma_j^2 + \sum_{m=1}^{M}\bar{h}_{mj}(n)p_m(n) + \sum_{j'\neq j}\bar{g}_{j'j}(n)p_{j'}^{\max}(n) + \epsilon_j(n)\sqrt{\sum_{j'\neq j}\frac{\left|p_{j'}^{\max}(n)\right|^2}{w_{j'j}(n)}} + \varepsilon_j(n)\sqrt{\sum_{m=1}^{M}\frac{|p_m(n)|^2}{w_{mj}(n)}},$$

$$d = \sigma_k^2 + \sum_{m=1}^{M}\bar{h}_{mk}(n)p_m(n) + \varepsilon_k(n)\sqrt{\sum_{m=1}^{M}\frac{|p_m(n)|^2}{w_{mk}(n)}}.$$

With the definition of Υ, a sufficient condition for the uniqueness of the RNE is provided in the following theorem.

Theorem 3.4: The follower sub-game admits a unique RNE if $\rho(\mathrm{I}-\Upsilon) < 1$, where $\rho(\cdot)$ is the spectral radius of a matrix.

Proof: See Appendix F. ■

Although not intuitive from Theorem 3.4, interference (including both co-tier and cross-tier interferences) impacts the uniqueness of the RNE. Specifically, the sufficient condition for uniqueness can be satisfied if the received or generated interference is low. Similar results are also shown in [5], [13], [16], and [21] for both nominal and robust power control games.

In the following, we propose an algorithm to arrive at the RNE. To this end, we first introduce the regularized gap function [23] of the formulated VI problem as

$$\Gamma(\mathbf{p}^{\text{low}}) = \sup_{\check{\mathbf{p}}^{\text{low}}\in\mathbf{X}}\Phi_\alpha(\mathbf{p}^{\text{low}},\check{\mathbf{p}}^{\text{low}}),$$

$$\text{where}\quad \Phi_\alpha(\mathbf{p}^{\text{low}},\check{\mathbf{p}}^{\text{low}}) = \mathbf{F}(\mathbf{p}^{\text{low}})(\mathbf{p}^{\text{low}}-\check{\mathbf{p}}^{\text{low}}) - \frac{\alpha}{2}\|\mathbf{p}^{\text{low}}-\check{\mathbf{p}}^{\text{low}}\|^2, \tag{20}$$

for a regularization parameter $\alpha > 0$.

Then we show that the VI reformulation of the follower GNEP leads to a fixed point problem as shown in the following proposition.

Proposition 3.5: $\mathbf{p}^{\text{low}*}$ is a solution of $VI(\mathbf{X},\mathbf{F})$ if and only if $\mathbf{p}^{\text{low}*}$ is a fixed point of the following equation:

$$\mathbf{p}^{\text{low}} = \arg\max_{\check{\mathbf{p}}^{\text{low}}\in\mathbf{X}}\Phi_\alpha(\mathbf{p}^{\text{low}},\check{\mathbf{p}}^{\text{low}}). \tag{21}$$

Based on proposition 3.5, the solution can be obtained by solving the fixed point problem for which the Picard iteration [26] based algorithm can be used as shown in **Algorithm** 1.

Algorithm 1 Obtaining the solution of the VI problem

---

1) Initialization: set $i=0$, select an initial power allocation $\mathbf{p}^{\text{low}}(i)$.
2) Update the power allocation as
$$\mathbf{p}^{\text{low}}(i+1) = \arg\max_{\check{\mathbf{p}}^{\text{low}}\in\mathbf{X}}\Phi_\alpha\left(\mathbf{p}^{\text{low}}(i),\check{\mathbf{p}}^{\text{low}}\right). \tag{22}$$
3) Repeat step 2 until termination condition is satisfied.

The above algorithm is a centralized one. To enable distributed solutions, we follow the penalty method in [21], [25]. In this method, the original problem is reformulated where a pricing-based penalty term is introduced into the payoff function of each player which is motivated by the appearance of the Lagrange multiplier of the global interference constraint (e.g., $\lambda_m(n)$ in (18)). Specifically, for each follower k, the reformulated problem is expressed as

$$\max_{\mathbf{p}_k}\min_{\mathbf{g}_k,\mathbf{h}_k,h_{km}(n)}\{\mathcal{R}_k(\mathbf{p}_k,\mathbf{p}_{-k},\mathbf{p}^{\text{up}}) - C_k(\mathbf{p}_k,\lambda)\} \tag{23}$$

$$\text{s.t.}\quad \mathbf{p}_k\in\bar{\mathcal{P}}_k,\ \mathbf{g}_k\in\mathcal{G}_k,\ \mathbf{h}_k\in\mathcal{H}_k,\ h_{km}(n)\in\mathcal{H}_{km}. \tag{24}$$

$C_k(\mathbf{p}_k,\lambda) = \sum_{n=1}^{N}\sum_{m=1}^{M}\lambda_m(n)h_{km}(n)p_k(n)$ is the penalty term. And the global interference constraint is taken into account by appropriately choosing the prices $\lambda_m = \{\lambda_m(n)\}_{n=1}^{N}$ to satisfy the following additional complementary conditions:

$$0 \leq \lambda_m(n)\ \perp\ I_m^{\max}(n) - \sum_{k=1}^{K}p_k(n)h_{km}(n) \geq 0, \tag{25}$$

where $a\perp b$ means $a\cdot b = 0$.

Due to the existence of the additional complementary conditions in (25), the analysis of the above reformulated game is challenging. However, the equivalence of the reformulated game and the $VI(\mathbf{X},\mathbf{F}(\mathbf{p}^{\text{low}}))$ can be established by showing the equivalence of the KKT conditions of the game and that of the VI problem (note that this reformulated game cannot be directly reformulated as a variational inequality problem due to the unboundedness of the prices). The previously obtained results on the existence and uniqueness can also be applied here. Although helpful in theoretical analysis, the reformulated game still results in a centralized solution as shown above. To enable distributed solutions, a nonlinear complementarity problem (NCP) equivalence can be established. To this end, we observe that the game has two parts: an inner game (23) which is a normal NEP and a complementarity condition. For the inner game, we have the following observations.

For a given price vector $\lambda = \{\lambda_m(n)\}_{m,n}$, the solution of the inner game exists. Under the condition that Υ is positive definite, the inner game admits a unique RNE. This existence and uniqueness can be simply shown by directly reformulating the inner game as a variational inequality problem $VI(\mathbf{X}^{\text{in}}, \mathbf{F}^{\text{in}}(\mathbf{p}^{\text{low}}))$, where $\mathbf{X}^{\text{in}} = \prod_{k=1}^{K} \bar{\mathcal{P}}_k$, $\mathbf{F}^{\text{in}}(\mathbf{p}^{\text{low}}) \triangleq \{-\nabla_{\mathbf{p}_k}[\hat{\mathcal{R}}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}}) - \hat{C}_k(\mathbf{p}_k, \lambda)]\}_{k=1}^{K}$, and $\hat{C}_k(\mathbf{p}_k, \lambda) = \sum_{n=1}^{N}\sum_{m=1}^{M} \lambda_m(n)(\bar{h}_{km}+\varepsilon_m(n)/\sqrt{w_{km}(n)})(n)p_k(n)$ is the worst-case valuation of the penalty term $C_k(\mathbf{p}_k, \lambda)$. Then a similar analysis could be done which is omitted here for brevity.

Denote by $\mathbf{p}_k^*(\lambda)$ the unique solution of the inner game for a given price vector λ (assume that the uniqueness condition is satisfied). We define a mapping $\Theta(\lambda)$ as

$$\Theta(\lambda) = \left\{ I_m^{\max}(n) - \sum_{k=1}^{K} \bar{h}_{km} p_k^* - \varepsilon_m(n) \sqrt{\sum_{k=1}^{K} p_k^2(n)} \right\}_{m\in\mathcal{M}, n\in\mathcal{N}}.$$

Then the game is equivalent to an $NCP(\Theta)$ problem given as

$$NCP(\Theta): 0 \le \lambda \perp \Theta(\lambda) \ge 0. \tag{26}$$

Based on the NCP reformulation, a distributed algorithm can be given as follows.

Algorithm 2 Algorithm for obtaining the optimal power allocation and price

---

1) Initialization: set $i = 0$, each MBS sets an initial price $\lambda_m^0(n), \forall n$, each follower k randomly chooses a power allocation $\mathbf{p}_k^0$.
2) Given the prices λ, each follower k computes its equilibrium power allocation as the solution of
$$\mathbf{p}_k^*\left(\lambda_m^i(n)\right) = \arg\max_{\mathbf{p}_k} \hat{\mathcal{R}}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}}) - \hat{C}_k(\mathbf{p}_k, \lambda).$$
3) The prices are updated by $\lambda_m^{i+1} = [\lambda_m^i - \tau\Theta_{mn}]^+$, where $\tau > 0$ is a chosen step size.
4) Repeat steps 2 and 3 until the termination condition is satisfied.

Remark: Note that price is also used in [9], [10]. However, different from these works, the price here is just used to control the interference, not for the purpose of profit maximization of the MBSs.

IV. Robust Leader Sub-Game

In this section, we provide a comprehensive analysis of the robust leader sub-game. Note that in the formulated robust Stackelberg game, the leaders need to anticipate the best responses of the followers in terms of the RNE of the follower sub-game. In this case, the leader sub-game becomes a robust equilibrium problem with equilibrium constraints (EPEC). And for each MBS m, the robust rate maximization problem becomes a mathematical program with equilibrium constraints (MPEC) which can be expressed as follows:

$$\max_{\mathbf{p}_m} \min_{\mathbf{g}_m, \mathbf{h}_m} \mathcal{R}_m\left(\mathbf{p}_m, \mathbf{p}_{-m}, \mathbf{p}^{\text{low}*}(\mathbf{p}^{\text{up}})\right) \tag{27}$$

$$\text{s.t.} \quad \mathbf{p}_m \in \mathcal{P}_m,\ \mathbf{g}_m \in \mathcal{G}_m,\ \mathbf{h}_m \in \mathcal{H}_m,$$

$$\mathbf{p}_k^*(\mathbf{p}^{\text{up}}) = \arg\max_{\mathbf{p}_k\in\mathcal{P}_k} \min_{\mathbf{g}_k\in\mathcal{G}_k, \mathbf{h}_k\in\mathcal{H}_k} \mathcal{R}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}_{\text{up}}),\ \forall k, \tag{28}$$

where $\mathcal{P}_m = \{\sum_{n=1}^{N} p_m(n) \le P_m, p_m(n) \ge 0\}$ is the power constraint for MBS m, and (28) is the equilibrium constraint. Note that we are considering service priority of the MBSs. In this case, the interference constraint will only be imposed for the SBSs, while there is no interference constraint for the MBSs.

Remark: Note that in general, the equilibrium response of the follower sub-game could be non-unique (i.e., multiple $\mathbf{p}^{\text{low}*}(\mathbf{p}^{\text{up}})$ could exist for a given $\mathbf{p}^{\text{up}}$), and each leader could have different estimations of the equilibrium response (i.e., there could exist $\mathbf{p}_m^{\text{low}*}(\mathbf{p}^{\text{up}}) \neq \mathbf{p}_l^{\text{low}*}(\mathbf{p}^{\text{up}})$ for $m \neq l$). In this case, the equilibrium constraint in (28) could be different for different MBSs. In this work, we assume that the uniqueness condition of the follower sub-game is satisfied (e.g., the sufficient conditions for uniqueness in [2]–[5] for nominal games, and those in [13]–[15] and the sufficient condition in this paper for robust games, are satisfied), and therefore, we have $\mathbf{p}_m^{\text{low}*}(\mathbf{p}^{\text{up}}) = \mathbf{p}^{\text{low}*}(\mathbf{p}^{\text{up}}), \forall m$.

Denote by $\mathbf{p}^{\text{up}*}$ an RNE of the leader sub-game, then the tuple $\{\mathbf{p}^{\text{up}*}, \mathbf{p}^{\text{low}*}(\mathbf{p}^{\text{up}*})\}$ constitutes a robust Stackelberg equilibrium (RSE) which is considered to be the solution of the robust Stackelberg game. In the following, the leader sub-game will be analyzed considering different formulations of the follower sub-game shown in Section III. Specifically, two cases will be considered: the follower sub-game with only power constraints and the follower sub-game with both power and global interference constraints. Note that the case of follower sub-game with both power and individual interference constraints can be equivalently transformed into a follower sub-game with only power constraints as shown in Section III, and therefore, will not be explicitly discussed here.

A. Follower Sub-Game With Only Power Constraints

In this part, we investigate the robust Stackelberg game with a follower sub-game considering only power constraints. Both column-wise and ellipsoidal uncertainty models are considered. Specifically, for the formulation with column-wise uncertainty model, two cases are further categorized, i.e., a sparse BS deployment scenario and a dense BS deployment scenario. For both cases, closed-form exact solutions can be obtained for a single leader. The results can be extended for multiple leaders if closed-form approximate solutions are considered. Also, sufficient conditions for the uniqueness of the closed-form solutions are provided for sparse BS deployment. For the formulation with ellipsoidal uncertainty model, a closed-form solution is not available. Instead, a distributed algorithm is provided to arrive at the RSE the existence of which can be established while the uniqueness cannot be guaranteed. The details are shown as follows.

1) With Column-Wise Uncertainty Model: We first consider the case with column-wise uncertainty model for which the uncertainty sets for the leader m can be expressed as

$$\mathcal{G}_m = \{\bar{g}_{lm}(n)+\triangle g_{lm}(n),\ |\triangle g_{lm}(n)| \le \epsilon_{lm}(n)\}_{l\neq m, l\in\mathcal{M}},$$
$$\mathcal{H}_m = \left\{\bar{h}_{km}(n)+\triangle h_{km}(n),\ |\triangle h_{km}(n)| \le \varepsilon_{km}(n)\right\}_{k\in\mathcal{K}}.$$

The uncertainty sets for the followers $\mathcal{G}_k$ and $\mathcal{H}_k$ are shown in Section III. Due to the difference in analysis, two cases are further categorized: sparse BS deployment and dense BS deployment scenarios.

i) Sparse BS deployment scenario: In this scenario, the interference experienced by SBSs is sufficiently small such that the following assumption can be satisfied.[1]

$$\mu_k - \tilde{v}_k(n) > 0, \quad \forall k,n, \tag{29}$$

where $\tilde{v}_k(n)$ is shown in (9). In this case, the followers are active in all subchannels.

We start the analysis by considering a single leader m. In this case, according to the analysis in Section III-A, the robust equilibrium power allocation of each follower k given the power allocation of the single leader $\mathbf{p}_m$ is obtained as

$$p_k^*(n) = \mu_k - \Big[\sigma_k^2(n) + \left(\bar{h}_{mk}(n)+\varepsilon_{mk}(n)\right)p_m(n) + \sum_{j\neq k}\left(\tilde{g}_{jk}(n)+\epsilon_{jk}(n)\right)p_j^*(n)\Big], \quad \forall k. \tag{30}$$

Accordingly, we can have a system of K linear equations as

$$\mathbf{A}\mathbf{p}^{\mathrm{low}*}(n)^{\mathrm{T}} = \mathbf{B}, \quad \text{where}$$
$$\mathbf{A} = \begin{bmatrix} 1 & \cdots & \tilde{g}_{K1}(n)+\epsilon_{K1}(n) \\ \vdots & \ddots & \vdots \\ \tilde{g}_{1K}(n)+\epsilon_{1K}(n) & \cdots & 1 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} \mu_1 - \sigma_1^2(n) - \left(\tilde{h}_{m1}(n)+\varepsilon_{m1}(n)\right)p_m(n) \\ \vdots \\ \mu_K - \sigma_K^2(n) - \left(\tilde{h}_{mK}(n)+\varepsilon_{mK}(n)\right)p_m(n) \end{bmatrix}. \tag{31}$$

Given the values of μ_k which can be numerically obtained by an iterative water-filling algorithm for the followers, the optimal power allocation of the followers can be represented as a linear function of the leader's power allocation as follows:

$$p_k^*(n) = a_k(n)p_m(n) + b_k(n), \tag{32}$$

where $a_k(n)$ and $b_k(n)$ are the corresponding coefficients which can be obtained by solving the above system of equations.

With the linear representation of the followers' best responses with respect to the leader's strategy, now let us turn to the robust optimization problem for a single leader which can be restated as

$$\max_{\mathbf{p}_m\in\mathcal{P}_m} \sum_{n=1}^{N} \log\left(1+\frac{p_m(n)}{a(n)p_m(n)+b(n)}\right), \tag{33}$$

where $a(n)=\sum_{k=1}^{K} a_k(n)(\bar{h}_{km}(n)+\varepsilon_{km}(n))$ and $b(n)=\sigma_m^2(n)+\sum_{k=1}^{K} b_k(n)(\bar{h}_{km}(n)+\varepsilon_{km}(n))$.

Applying the KKT conditions, it is not difficult to find the unique optimal solution of the single leader's power allocation as follows (the derivation is omitted here for brevity):

$$p_m^*(n) = -\frac{b(n)\,(1+2a(n))}{2a(n)\,(1+a(n))}. \tag{34}$$

Accordingly, the tuple $\{\mathbf{p}_m^*, \mathbf{p}^{\mathrm{low}*}\}$ constitutes the RSE with $\mathbf{p}_k^* = a_k(n)p_m^*(n)+b_k(n)$.

Regarding the existence and uniqueness of the RSE, we have shown that for any given leader's strategy, the follower sub-game admits a unique equilibrium if certain condition is satisfied. Then the problem is to determine whether the optimal strategy for the leader exists and is unique given the unique response of the followers which in turn relies on the existence and uniqueness of the coefficients $a_k(n)$ and $b_k(n)$. A sufficient condition for the uniqueness of the RSE is provided in the following theorem.

Theorem 4.1: With a sparse BS deployment where (29) is satisfied, the single-leader multiple-follower robust Stackelberg game with column-wise uncertainty model admits a unique RSE if $rank(A) = K$.

The proof is straightforward in that if $rank(A) = K$ then the system of linear equations admits a unique solution. It is worth noting that this uniqueness condition can almost be always satisfied in practice due to the random nature of the channel gain which makes all K coefficient vectors independent of each other, and therefore, achieving the full rank.

We now extend the analysis for the multiple-leader multiple-follower formulation. Similar to the analysis of the single leader case, the best response power allocation of the followers given the power allocations of all leaders is the solution of the following system of equations:

$$\mathbf{A}\mathbf{p}^{\mathrm{low}*}(n)^{\mathrm{T}} = \tilde{\mathbf{B}}, \quad \text{where}$$
$$\tilde{\mathbf{B}} = \begin{bmatrix} \mu_1 - \sigma_1^2(n) - \sum_{m=1}^{M}\left(\tilde{h}_{m1}(n)+\varepsilon_{m1}(n)\right)p_m(n) \\ \vdots \\ \mu_K - \sigma_K^2(n) - \sum_{m=1}^{M}\left(\tilde{h}_{mK}(n)+\varepsilon_{mK}(n)\right)p_m(n) \end{bmatrix}. \tag{35}$$

Then by solving the above system of equations, we can have

$$p_k^*(n) = \sum_{m=1}^{M} a_{km}(n)p_m(n) + \tilde{b}_k(n), \tag{36}$$

where $a_{km}(n)$ and $b_k(n)$ are the corresponding coefficients. In other words, the power allocation of each follower can be represented by a linear combination of power allocations of all leaders.

[1]The cases, in which this assumption does not hold in a sparse BS deployment scenario, can be analyzed in the same way as that in a dense BS deployment scenario.

However, different from the single leader case, in the multiple-leader formulation, the robust optimization for the leaders' problems will induce a system of quadratic equations which would be not tractable. In this case, instead of obtaining the exact solution, we aim to find a closed-form approximate solution. To this end, when optimizing the leader's rate, we assume that the values of optimal response from the followers with respect to the leaders' optimal strategies (i.e., $p_k^*(p_m^*)$) are given and are treated as constants, then we substitute the exact linear representation of the followers' strategy (i.e., (36)) into the obtained leaders' strategies. With this assumption, we can have the optimal power allocation of the leader m as

$$p_m^*(n) = \mu_m - \sigma_m^2(n) - \sum_{m\neq l} \left(\bar{g}_{lm}(n) + \epsilon_{lm}(n)\right) p_l^*(n) - \sum_{k=1}^{K} \left(\bar{h}_{km}(n) + \varepsilon_{km}(n)\right) p_k^*(n, p_m^*). \quad (37)$$

Substituting (36) into (37), we can have a system of M linear equations as

$$\hat{\mathbf{A}}\mathbf{p}^{\mathrm{up}*}(n)^{\mathrm{T}} = \hat{\mathbf{B}}, \quad (38)$$

where $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ can be constructed accordingly. And the closed-form approximate solution for the leaders can be obtained by solving this system of equations. Also, a similar uniqueness condition is provided in the following theorem.

Theorem 4.2: With a sparse BS deployment where (29) is satisfied, the multiple-leader multiple-follower robust Stackelberg game with column-wise uncertainty model admits a unique closed-form approximate RSE if $rank(\hat{\mathbf{A}}) = M$ and $rank(\mathbf{A}) = K$.

ii) Dense BS deployment scenario: In the case of dense BS deployment, the assumption in (29) may not hold. In this case, results similar to those in [8] can be obtained. However, in [8], the spectral mask constraints of the followers are not considered. Here we extend the analysis by considering the spectral mask constraints.

We also start the analysis with a single-leader multiple-follower formulation. Denote by $\mathcal{A}_m$ and $\mathcal{A}_k$ the sets of active subchannels[2] of the single leader m and follower k, respectively. Also, denote by $\mathcal{A}_k^{\max}$ the set of subchannels to which follower k allocates its maximum allowable power. Apparently, we have $\mathcal{A}_k^{\max} \subseteq \mathcal{A}_k$. With the spectral mask constraints, the best response power allocation of follower k is given by

$$p_k(n) = \begin{cases} 0, & n \in \mathcal{N} \setminus \mathcal{A}_k, \\ \mu_k - \tilde{\nu}_k(n), & n \in \mathcal{A}_k \setminus \mathcal{A}_k^{\max}, \\ p_k^{\max}(n), & n \in \mathcal{A}_k^{\max}, \end{cases} \quad (39)$$

where $\mu_k = \frac{1}{|\mathcal{A}_k \setminus \mathcal{A}_k^{\max}|}\left(P_k^{\mathrm{sum}} - \sum_{n\in\mathcal{A}_k^{\max}} p_k^{\max}(n) + \sum_{n\in\mathcal{A}_k\setminus\mathcal{A}_k^{\max}} \tilde{\nu}_k(n)\right)$.

Now we represent the followers' optimal power allocations for a given leader's strategy. For a subchannel n, let us denote by $\mathcal{J}(n)$ the set of interfering followers for the leader. Also, let $\mathcal{A}^{\max}(n)$ denote the set of followers which allocate maximum power on subchannel n. Then the power allocated by follower k, where $k \notin \mathcal{A}^{\max}(n)$, can be represented by

$$p_k^*(n) = \mu_k - \left(\sigma_k^2(n) + \left(\bar{h}_{mk}(n) + \varepsilon_{mk}(n)\right) p_m(n) + \left(\bar{g}_{jk}(n) + \epsilon_{jk}(n)\right) \left(\sum_{j\in\mathcal{J}(n)\setminus\mathcal{A}^{\max}(n)} p_j^*(n) + \sum_{j\in\mathcal{A}^{\max}(n)} p_j^{\max}(n) \right)\right).$$

Accordingly, we can have a system of $|\mathcal{J}(n) \setminus \mathcal{A}^{\max}(n)|$ linear equations the solution of which can be denoted by

$$p_k^*(n) = \tilde{a}_k(n) p_m(n) + \tilde{b}_k(n),\ \forall k \in \mathcal{J}(n) \setminus \mathcal{A}^{\max}(n), \quad (40)$$

where $\tilde{a}_k(n)$ and $\tilde{b}_k(n)$ are the corresponding coefficients.

Given the active sets of $\mathcal{A}_k$ and $\mathcal{A}_k^{\max}$, denote by $I = \mathcal{A}_m \bigcap (\cup_{k\in\mathcal{K}} \mathcal{A}_k)$ the set of interfering subchannels for the leader. Then by substituting (40), the leader's robust rate maximization problem can be restated as

$$\max_{\mathbf{p}_m} \quad \sum_{n\notin\cup_{k\in\mathcal{K}}\mathcal{A}_k} \log\left(1 + \frac{p_m(n)}{\sigma_m^2(n)}\right) + \sum_{n\in\cup_{k\in\mathcal{K}}\mathcal{A}_k} \log\left(1 + \frac{p_m(n)}{\tilde{a}_n p_m(n) + \tilde{b}(n)}\right), \quad (41)$$

where $\tilde{a}_n = \sum_{k\in\mathcal{J}(n)\setminus\mathcal{A}^{\max}(n)} \tilde{a}_k(n)(\bar{h}_{km}(n) + \varepsilon_{km}(n))$ and $\tilde{b}_n = \sigma_m^2(n) + \sum_{k\in\mathcal{J}(n)\setminus\mathcal{A}^{\max}(n)}(\bar{h}_{km}(n) + \varepsilon_{km}(n))\tilde{b}_k(n) + \sum_{k\in\mathcal{A}^{\max}(n)}(\bar{h}_{km}(n) + \varepsilon_{km}(n))p_k^{\max}(n)$.

By applying the KKT conditions, the optimal power allocation for the single leader can be obtained as

$$p_m(n) = \begin{cases} 0, & n \in \mathcal{N} \setminus \mathcal{A}_m, \\ \mu_m - \sigma_m^2(n), & n \in \mathcal{A}_m \setminus I, \\ -\frac{\tilde{b}(n)(1+2\tilde{a}(n))}{2\tilde{a}(n)(1+\tilde{a}(n))}, & n \in I. \end{cases} \quad (42)$$

Similar to the sparse deployment scenario, for the multiple-leader formulation, we also try to find a closed-form approximate solution. Under the same assumption, for each leader m we have

$$p_m^*(n) = \mu_m - \sigma_m^2(n) - \sum_{m\neq l} \left(\bar{g}_{lm}(n) + \epsilon_{lm}(n)\right) p_l^*(n) - \sum_{k=1}^{K} \left(\bar{h}_{km}(n) + \varepsilon_{km}(n)\right) p_k^*\left(n, p_m^*\right), \quad (43)$$

where

$$\sum_{k=1}^{K} \left(\bar{h}_{km}(n) + \varepsilon_{km}(n)\right) p_k^*\left(n, p_m^*\right) = \left(\bar{h}_{km}(n) + \varepsilon_{km}(n)\right) \times \left(\sum_{k\in\mathcal{J}\setminus\mathcal{A}^{\max}} (n) p_k^*\left(n, p_m^*\right) + \sum_{k\in\mathcal{A}^{\max}} p_k^{\max}(n) \right),$$

and $p_k^*(n, p_m^*) = \tilde{a}_k(n) p_m^*(n) + \tilde{b}_k(n)$. Accordingly, we can also have a system of M linear equations and the closed-form approximate solution can be obtained by solving this system of equations.

[2] An active subchannel means a player allocates some non-zero power in this subchannel.

2) With Ellipsoidal Uncertainty Model: The uncertainty sets for leader m are expressed as

$$\mathcal{G}_m = \left\{ \bar{g}_{lm}(n) + \triangle g_{lm}(n), \|\triangle g_{lm}(n)\|_{\mathbf{w}_{lm}} \le \epsilon_m^2(n) \right\}_{l \neq m, l \in \mathcal{M}}.$$

The uncertainty sets for the followers are shown in Section III. According to the analysis in Section III, the robust equilibrium power allocation for each follower k is a robust water-filling like solution given as $\mathbf{p}_k^* = [\mu_k - \hat{v}_k(n)]^+$ with only total power constraint or $\mathbf{p}_k^* = [\mu_k - \hat{v}_k(n)]_0^{p_k^{\max}(n)}$ if the spectral mask constraints of the followers are considered. And

$$\hat{v}_k(n) = \sigma_k^2(n) + \sum_{m=1}^{M} \bar{h}_{mk}(n) p_m(n) + \sum_{j \neq k} \bar{g}_{jk}(n) p_j(n) + \epsilon_k(n) \sqrt{\sum_{j \neq k} \frac{|p_j(n)|^2}{w_{jk}(n)}} + \varepsilon_k(n) \sqrt{\sum_{m=1}^{M} \frac{|p_m(n)|^2}{w_{mk}(n)}}.$$

We first show the existence of a RSE in the following theorem.

Theorem 4.3: There exists at least one robust Stackelberg equilibrium for the formulated multiple-leader multiple-follower robust Stackelberg game with ellipsoidal uncertainty model.

Proof: See Appendix G. ■

However, due to the nonlinear terms in the followers' best response, a linear representation of the followers' best response strategies with respect to the leaders' strategies is not available which is different from the column-wise uncertainty case. Accordingly, a closed-form solution for the leader cannot be obtained for either of the sparse or dense deployment scenarios, and therefore, we do not distinguish them in this formulation. Specifically, given the leaders' strategies $\mathbf{p}^{\text{up}}$, the followers respond with the unique optimal power allocation $\mathbf{p}^{\text{low}*}$ (assuming that the uniqueness condition of the follower game is satisfied), which can be numerically obtained. Then each leader m updates its strategy according to the following equation:

$$p_m(n) = \mu_m - \hat{v}_m(n), \tag{44}$$

where

$$\hat{v}_m(n) = \sigma_m^2(n) + \sum_{l \neq m} \bar{g}_{lm} p_l(n) + \sum_{k=1}^{K} \bar{h}_{km} p_k^*(n) + \epsilon_m(n) \sqrt{\sum_{l \neq m} \frac{|p_l(n)|^2}{w_{lm}(n)}} + \varepsilon_m(n) \sqrt{\sum_{k=1}^{K} \frac{|p_k^*(n)|^2}{w_{km}(n)}}.$$

Accordingly, an algorithm to arrive at the RSE of the game is proposed as shown in **Algorithm 3**.

Algorithm 3 Algorithm for reaching the RSE

---

1) Initialization: set $i = 0$, each MBS chooses an initial power allocation $\mathbf{p}_m^i$.
2) Given the power allocation of all MBSs $\mathbf{p}^{\text{up}}(i)$, each SBS k responds with the unique power allocation $\mathbf{p}_k^*(i)$.
3) With the best response power allocations of the followers, and given the power allocations of other MBSs, each MBS m updates its power as $p_m^{i+1}(n) = \mu_m - \hat{v}_m^i(n)$.
4) Repeat step 2 and 3 until termination condition is satisfied.

B. Follower Sub-Game With Power and Global Interference Constraints

We investigate the robust Stackelberg game formulation with follower sub-game considering both power and global interference constraints. Note that the interference constraints are not imposed to MBSs, and the leader sub-game only considers power constraints. Also, we consider both uncertainty models.

1) With Column-Wise Uncertainty Model: For the robust follower sub-game with column-wise uncertainty model, the best response power allocation for each follower k is

$$p_k^*(n) = \left[\frac{1}{1/\mu_k + \sum_{m=1}^{M} \lambda_m(n) \left(\bar{h}_{km}(n) + \varepsilon_{km}(n) \right)} - \tilde{v}_k(n) \right]^+,$$

where $\lambda_m(n), \forall m, n$ are the Lagrange multipliers associated with the global interference constraints which can also be interpreted as prices.

Given the values of μ_k and $\lambda_m(n)$, the followers' strategies can be represented by a linear combination of leaders' strategies. And the following analysis is similar to the case of follower game with power constraints with the replacement of μ_k with $\frac{1}{1/\mu_k + \sum_{m=1}^{M} \lambda_m(n)(\bar{h}_{km}(n) + \varepsilon_{km}(n))}$. Accordingly, for both sparse and dense BS deployment scenarios, exact closed-form solutions can be obtained for the single-leader multiple-follower formulation and closed-form approximate solutions can be obtained for the multiple-leader multiple-follower formulation. Also, similar uniqueness conditions can be obtained. The details are omitted here for brevity.

2) With Ellipsoidal Uncertainty Model: With ellipsoidal uncertainty model, the best response strategies of the followers cannot be obtained in closed form. Accordingly, a closed-form RSE of the robust Stackelberg game is not available. However, the existence of a RSE can still be established in the following theorem.

Theorem 4.4: For the formulated multiple-leader multiple-follower robust Stackelberg game with ellipsoidal uncertainty model, in which the follower sub-game considers both power constraints and global interference constraints, there exists at least one RSE.

The proof is similar to that of Theorem 4.3. For numerically obtaining the RSE, an algorithm is proposed. Specifically, the Lagrange multipliers $\lambda_m, \forall m$ are considered as prices which are used by the MBSs solely for controlling the interference purpose. Then given the interference prices and leaders' power allocations, the followers respond with unique equilibrium power allocation with which each MBS updates its prices and power allocation.

Algorithm 4 Algorithm for obtaining leaders' optimal power allocation and price

---

1) Initialization: set $i = 0$, each MBS m sets an initial price λ_m^i and an initial power allocation $\mathbf{p}_m^i$.
2) Given the prices λ_m^i and power allocation $\mathbf{p}_m^i$ of all M MBSs, each SBS k responds with $\mathbf{p}^{i*}(\lambda_m^i, \mathbf{p}_m^i)$ which can be obtained from **Algorithm 2**.
3) Then each MBS m updates its price as $\lambda_m^{(i+1)} = [\lambda_m^{(i)} - \tau\Theta_{mn}]^+$.
4) Each MBS m updates its power as $p_m(n) = \mu_m - \hat{\nu}_m^i(n)$.
5) Repeat step 2 and 4 until the termination condition is satisfied.

V. Numerical Analysis

A. Parameters

For the numerical analysis, we consider a two-tier HetNet consisting of $M = 2$ MBSs and $K = 5$ SBSs sharing $N = 10$ orthogonal subchannels. For each subchannel, the noise power at the receiver is -120 dBm. Independent and identically distributed Rayleigh fading model is assumed for each subchannel. Accordingly, the channel gain can be obtained by squaring the independent Rayleigh random variable which is described by the modulus of circularly symmetrical complex Gaussian (CSCG) distribution. The average nominal direct channel gains for the MBSs and SBSs are assumed to be 1 and 3, respectively. For the uncertainty model, we assume that the uncertainty term varies linearly with the nominal value. Specifically, we use $g_{lm}(n)$ as an example, where $g_{lm}(n) = \bar{g}_{lm}(n) + \triangle g_{lm}(n)$ with $\triangle g_{lm}(n) = \theta \bar{g}_{lm}(n)$ and $\theta \sim U(-\delta, \delta)$. Accordingly, we can have the uncertainty bound for column-wise model as $\epsilon_{lm}(n) = \delta \bar{g}_{lm}(n)$ and the uncertainty bound for ellipsoidal model as $\epsilon_m(n) = \delta \|\bar{g}_{lm}(n)\|_{\mathbf{w}_{lm}(n)}$.

B. Results

We first show the convergence of the algorithm for obtaining an RSE of the formulated RSG with ellipsoidal uncertainty model. Specifically, the leaders and the followers iteratively update the power allocations and converge to the RSE. The evolution of rates for the leaders and followers is shown in Figs. 1 and 2, respectively. Note that due to the larger number of players, the convergence of the followers is slower than that of the leaders.

We also investigate the impact of interference on the convergence (Fig. 3). We can observe that when the interference level is low, the convergence can be guaranteed. With the increase of interference level, the convergence probability decreases. The convergence probability decreases even further when the numbe of players increases (e.g., from 5 SBSs to 20 SBSs).

To compare the robust solutions with the nominal solution, we first consider the formulation with only power constraints. The robust solutions under column-wise uncertainty model and ellipsoidal uncertainty model are obtained using the cor-

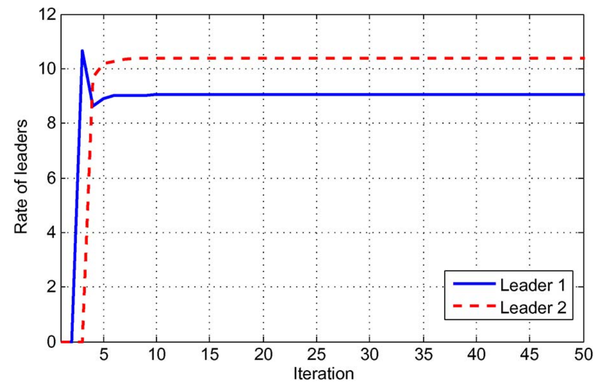

Fig. 1. The convergence of rate of leaders.

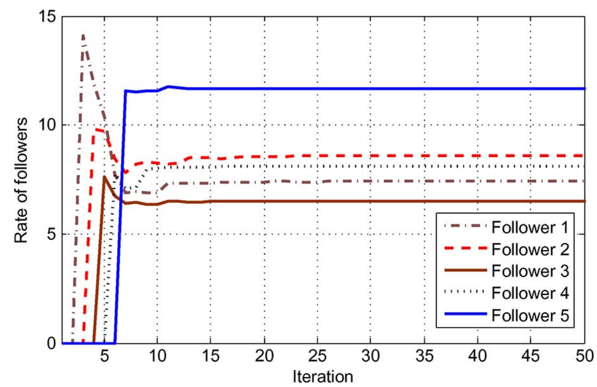

Fig. 2. The convergence of rate of followers.

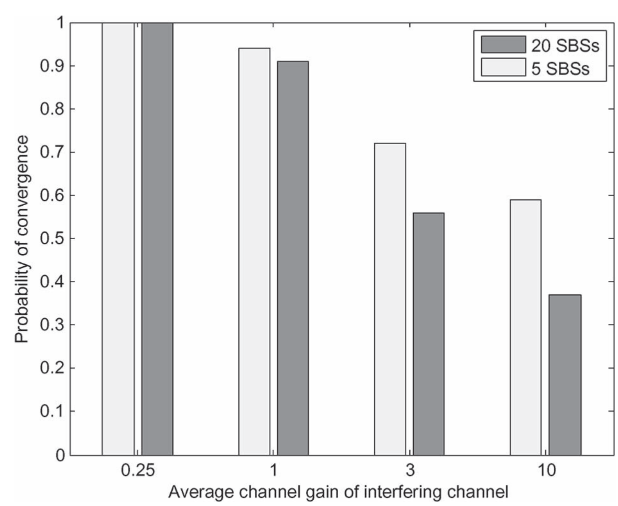

Fig. 3. Convergence probability under different interference level.

responding uncertainty bounds, and the nominal solution is obtained using the nominal value of the channel gains. The performances of these three solutions are evaluated in a scenario where the CSI is imperfect. That is, the actual value of the channel gain deviates from the nominal value with a uncertainty term within the uncertainty bound. The average rates of the leaders and the followers are shown in Figs. 4 and 5, respectively. We can observe that with uncertainty in channel information, both of the robust solutions offer better performances than the nominal solution for both the leaders and the followers. The performance gap increases with an

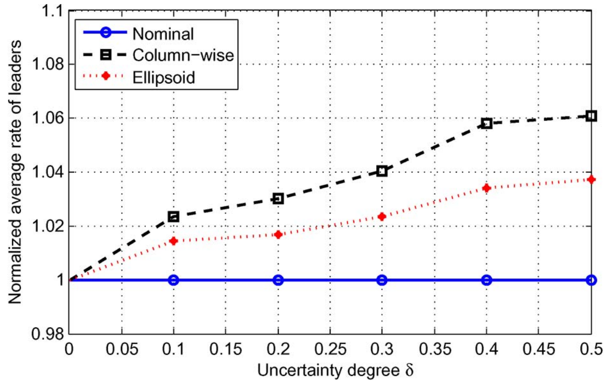

Fig. 4. The normalized average rate of leaders.

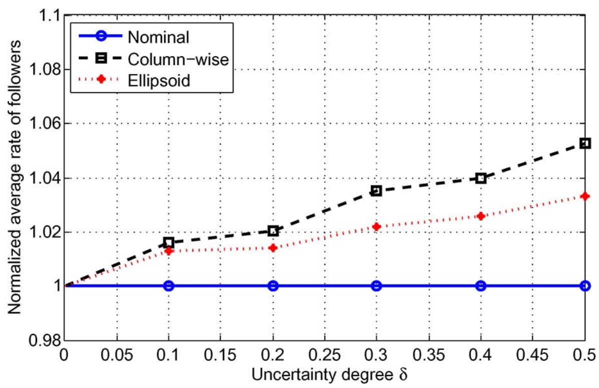

Fig. 5. The normalized average rate of followers.

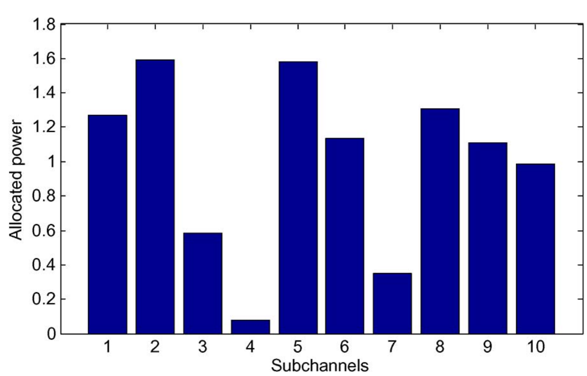

Fig. 6. Equilibrium power allocation of leader 1 in nominal formulation.

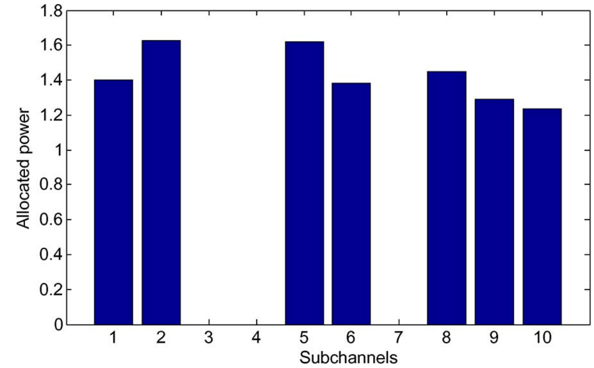

Fig. 7. Equilibrium power allocation of leader 1 in robust formulation.

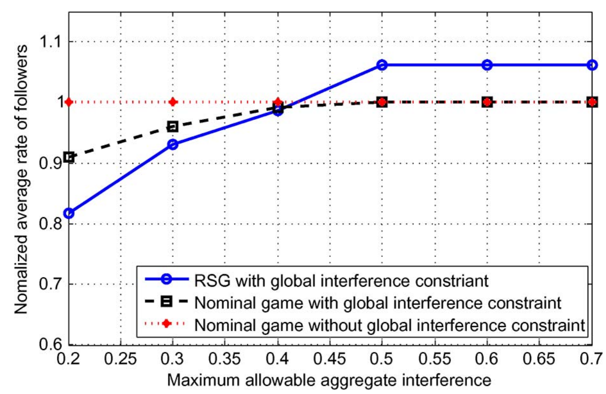

Fig. 8. Normalized average rate of followers under different level of aggregate interference constraints.

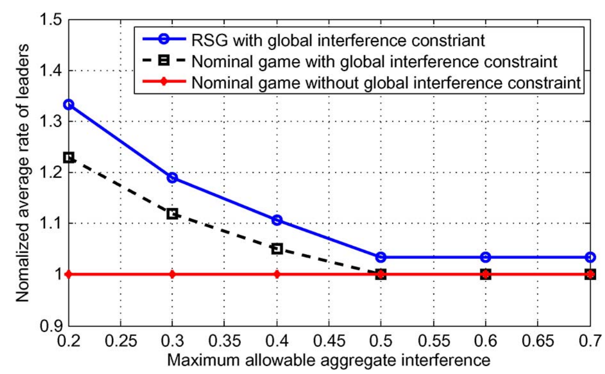

Fig. 9. Normalized average rate of leaders under different level of aggregate interference constraints.

increase of the degree of uncertainty. This is due to the fact that with the worst-case robust solution, the power allocation is more conservative, and less power is allocated for high interfering channels when compared with the nominal solution. Examples of the power allocations of leader 1 with the nominal solution and the robust solution are shown in Figs. 6 and 7, respectively. Also, due to the more conservative nature, in an imperfect CSI environment, the solution based on the column-wise uncertainty model performs better than that based on the ellipsoidal uncertainty model.

We then consider the formulation with global interference constraints. Specifically, three solutions are compared, i.e., a robust solution for the RSG considering global interference constraints, a nominal solution for the nominal SG considering global interference constraints, and a nominal solution for the nominal SG without considering global interference constraints. The performances of these three solutions under different aggregate interference constraints are shown in Figs. 8 and 9 for the followers and the leaders, respectively. We can observe that for the followers, with strong aggregate interference constraints, the solution without considering global interference constraints performs better than the others. This is due to the fact that, the interference constraints are imposed at the follower side, and for satisfying the strong interference constraints, only portion of the total power can be allocated. The solution without considering global interference constraints can allocate its whole power, and therefore, results in a better performance. The robust solution performs the worst due to the requirement of satisfying the interference constraint under worst-case

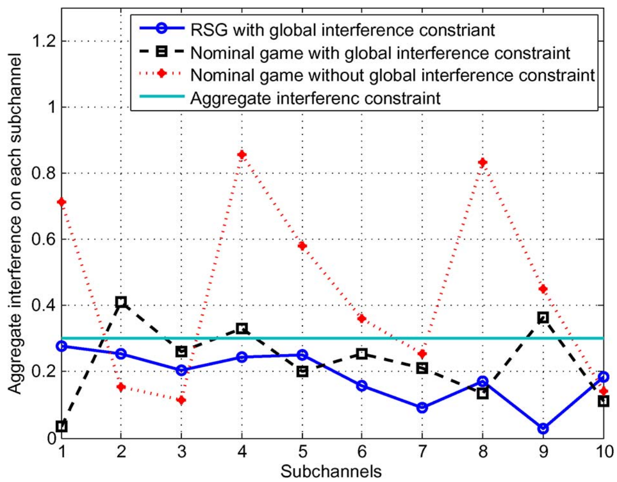

Fig. 10. The aggregate interference experienced by a user of an MBS under different solutions.

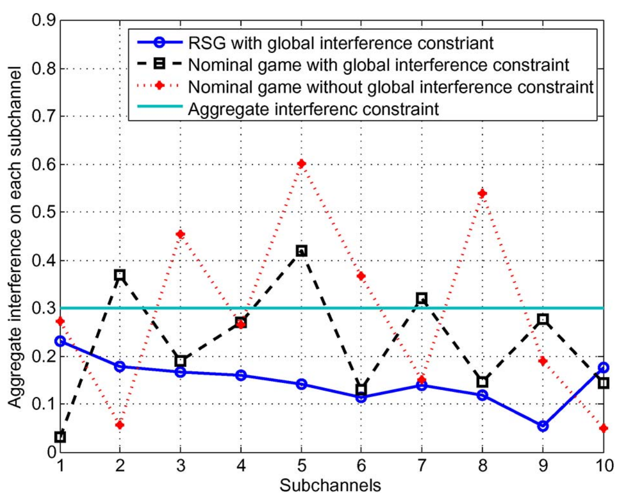

Fig. 11. The aggregate interference experienced by a user of an MBS under different solutions.

channel realizations. When the global interference constraints are loose enough to be inactive, then only the power constraints are active, and accordingly, the robust solution performs better than the others in such an imperfect CSI environment. For the leaders, we can observe that the solution without considering the global interference constraints performs the worst. The reason is that the leaders are not protected in this case.

In Figs. 10 and 11, we show the interference experienced by the leaders in each subchannel. We can observe that the interference constraint could be violated if nominal solutions are used in an imperfect CSI environment. With the robust solutions, the interference constraints can be satisfied for each subchannel.

VI. Conclusion

The problem of robust downlink power control for rate maximization in OFDMA-based HetNets has been investigated considering uncertainty in interfering channel information. Specifically, the hierarchical interactions among the MBSs and SBSs have been modeled as a multi-leader multi-follower robust Stackelberg game where the MBSs and SBSs are considered to be the leaders and the followers, respectively. A comprehensive treatment of the robust Stackelberg game has been provided considering various power and interference constraints. Also, both column-wise and ellipsoidal uncertainty models have been considered. Robust Stackelberg equilibrium (RSE) has been obtained as the solution of the robust Stackelberg game and the existence of the solution has been shown. In certain cases, a closed-form RSE has been obtained, and mild sufficient conditions for its uniqueness have been provided. Algorithms have been proposed for obtaining the RSE. Also, numerical analysis has been performed which shows the effectiveness of the robust solutions.

Appendix

A. Stackelberg Game

Stackelberg game [30], also known as the leader-follower game, is featured with a hierarchical structure and non-simultaneous moves of players. In this hierarchical structure, the players are grouped as leaders and followers. Different from the simultaneous play non-cooperative games in which all players make their moves simultaneously, in a Stackelberg game, the leaders make their moves before the followers. Note that the followers can observe the strategies made by the leaders and make their best response moves accordingly. The leaders are aware of this and can anticipate the best responses of the followers which they take into account when making their moves. Stackelberg equilibrium is considered to be the solution of a Sackelberg game at which none of the leaders or followers can improve the payoff by unilaterally changing the strategy.

B. Variational Inequality

Variational inequality (VI) [22] is a discipline in the field of mathematical programming which provides a unifying framework to study optimization and equilibrium problems. The definition of a VI problem is as follows:

Given a subset $\mathcal{K}$ of the Euclidean n-dimensional space $\mathbb{R}^n$ and a mapping $\mathbf{F}: \mathcal{K} \to \mathbb{R}^n$, the VI problem $\mathtt{VI}(\mathcal{K}, \mathbf{F})$ is to find a vector $\mathbf{x}^* \in \mathcal{K}$ such that

$$(\mathbf{x} - \mathbf{x}^*)^T \mathbf{F}(\mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in \mathcal{K}.$$

Problems such as fixed point problems and game theory problems can be formulated as VI problems.

C. Robust Optimization and Worst-Case Approach

Traditional optimization problems (i.e., nominal optimization problems) are based on the important assumption that all parameters defining the optimization problems (including both objective functions and constraints) can be precisely obtained. However, uncertainties exist for many parameters in practice. In such a case, the solutions obtained from solving nominal optimization problems may lead to poor performance in a practical imperfect information environment. To this end, robust optimization [31] has been developed to deal with optimization problems with parameter uncertainties. In general, robust optimization can be classified into "probabilistic" Bayesian based approach and "non-probabilistic" worst-case based approach. With the Bayesian approach, the uncertain parameters are considered as random variables with known probability distributions, and the performance is optimized on an average basis (i.e., optimizing the expectation of the objective function). With the worst-case approach, the uncertain parameters are considered to be bounded within certain uncertainty regions, and the performance is optimized for the worst cases (i.e., optimizing the possible worst parameter realizations within the uncertainty region) for which "Wald's maxmin model" is usually employed, i.e.,

$$\max_x \min_u f(x,u),$$

where the decision maker tries to maximize the objective function $f(x,u)$ by controlling the decision variable x, while the nature (i.e., representing the uncertainty) tries to minimize the objective function by finding the worst-case uncertain parameter realization within the uncertaity bound.

D. Proof of Lemma 3.1

The proof is based on Theorem 3.3 of [20] which requires the objective function and admissible set satisfy the convexity assumption for proving the equivalence of a GNEP and a QVI problem.

Considering the ellipsoidal uncertainty model, for each follower k, the worst-case analysis of the payoff function $\hat{\mathcal{R}}_k(\mathbf{p}_k, \mathbf{p}_{-k}, \mathbf{p}^{\text{up}})$ is continuously differentiable and is convex with respect to $\mathbf{p}_k$ for any given $\mathbf{p}_{-k}$. Also, the admissible set $\bar{\mathcal{P}}_k \bigcap \hat{\mathcal{P}}_k(\mathbf{p}_{-k})$ is closed and convex with given $\mathbf{p}_{-k}$. In this case, the convexity requirement in Theorem 3.3 of [20] is satisfied. Accordingly, the equivalence can be established. Specifically, $\mathbf{p}^{\text{low}*}$ is a GNE if and only if $\mathbf{p}^{\text{low}*}$ solves the $QVI(\mathbf{X}(\mathbf{p}^{\text{low}}), \mathbf{F}(\mathbf{p}^{\text{low}}))$.

E. Proof of Theorem 3.2

The proof is based on the equivalence of the GNEP and the VI problem. Specifically, we first show that the solution of $VI(\mathbf{X}, \mathbf{F}(\mathbf{p}^{\text{low}}))$ exists since the set $\mathbf{X}$ is convex and compact (i.e., closed and bounded) and the function $\mathbf{F}(\mathbf{p}^{\text{low}})$ is continuous in $\mathbf{p}^{\text{low}}$[22]. Then based on the equivalence, we can state that the RNE exists for the follower sub-game with global interference constraint.

F. Proof of Theorem 3.4

The proof is based on Lemma 3.3 which states that "The $VI(\mathbf{X}, \mathbf{F})$ admits a unique solution if $\mathbf{F}$ is strongly monotone." In the following, we give the main steps for the proof. First, if the matrix Υ is positive definite, the vector function $\mathbf{F}$ is strongly monotone. Then the $VI(\mathbf{X}, \mathbf{F}(\mathbf{p}^{\text{low}}))$ admits a unique solution. A sufficient condition for Υ to be positive definite is $\rho(\mathrm{I} - \Upsilon) < 1$. Based on the equivalence of the GNEP and the VI problem, we can have the uniqueness condition for the GNEP. Note that due to space limitation, the lengthy intermediate steps of the proof are omitted.

G. Existence of RSE

The proof is based on Schauder fixed point theory, which states that every continuous function from a convex compact subset to itself has a fixed point.

Denote by BR^{up} and BR^{low} the best response functions for the leader sub-game and the follower sub-game, respectively. Accordingly, an RNE $\mathbf{p}^{\text{up}*}$ of the leader sub-game exists if it is a fixed point of

$$\mathbf{p}^{\text{up}*} = BR^{\text{up}}\left(\mathbf{p}^{\text{up}*}, BR^{\text{low}}(\mathbf{p}^{\text{up}*})\right).$$

Since $BR^{\text{low}}(\mathbf{p}^{\text{up}*})$ is a continuous function of $\mathbf{p}^{\text{up}*}$, and BR^{up} is a continuous function of $\mathbf{p}^{\text{up}*}$ and $BR^{\text{low}}(\mathbf{p}^{\text{up}*})$, then BR^{up} is a continuous function of $\mathbf{p}^{\text{up}*}$. Also, the admissible set $\mathcal{P}^{\text{up}}$ is convex and compact. According to Schauder fixed point theory, at least one RNE of the leader sub-game exists. Also, we have shown in Section III that the RNE of the lower sub-game exists for any given leaders' power allocation. Therefore, at least one RNE $\{\mathbf{p}^{\text{up}*}, \mathbf{p}^{\text{low}*}(\mathbf{p}^{\text{up}*})\}$ exists.

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